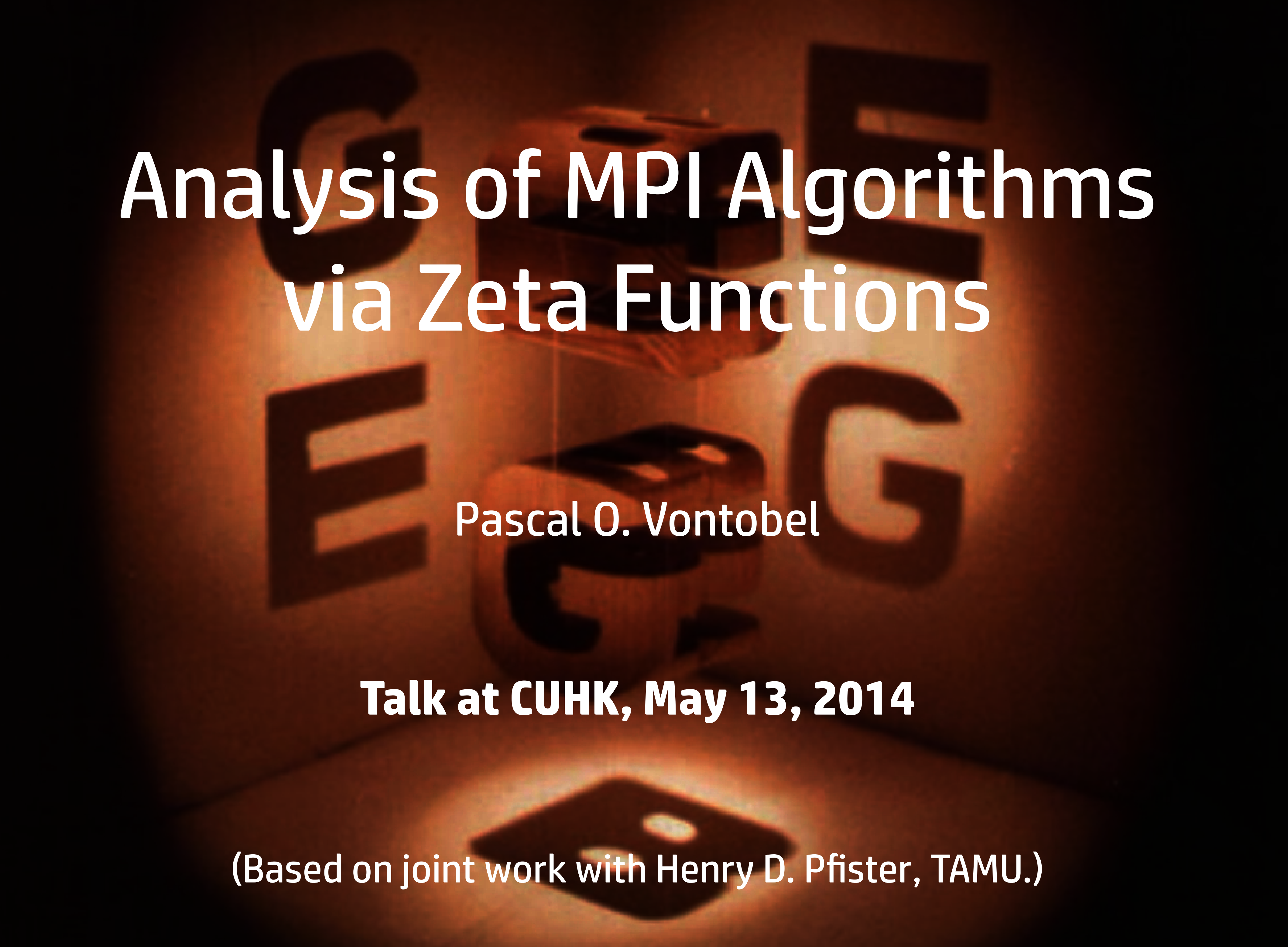

Analysis of MPI Algorithms via Zeta Functions

Pascal O. Vontobel

Talk at CUHK, May 13, 2014

(Based on joint work with Henry D. Pfister, TAMU.)

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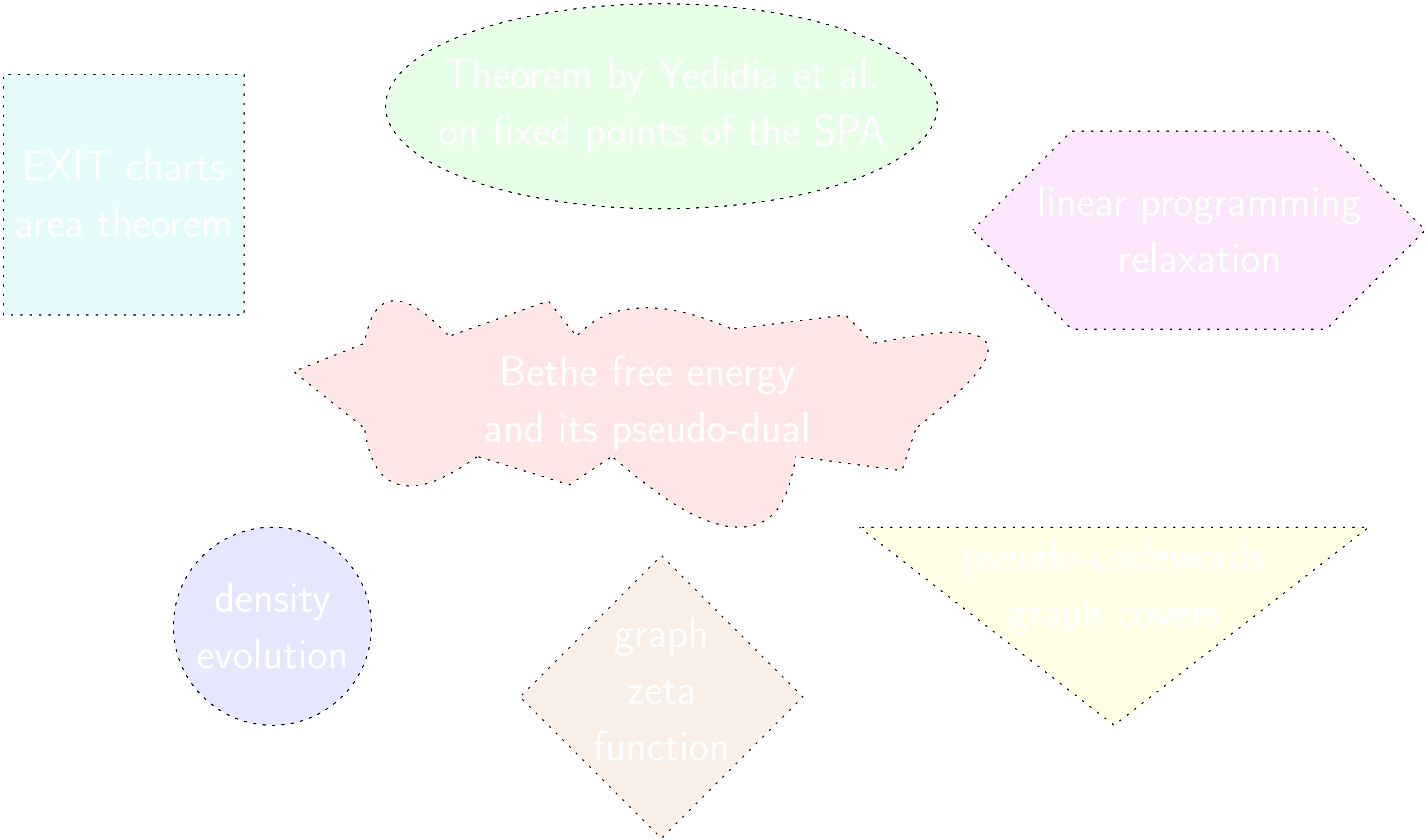
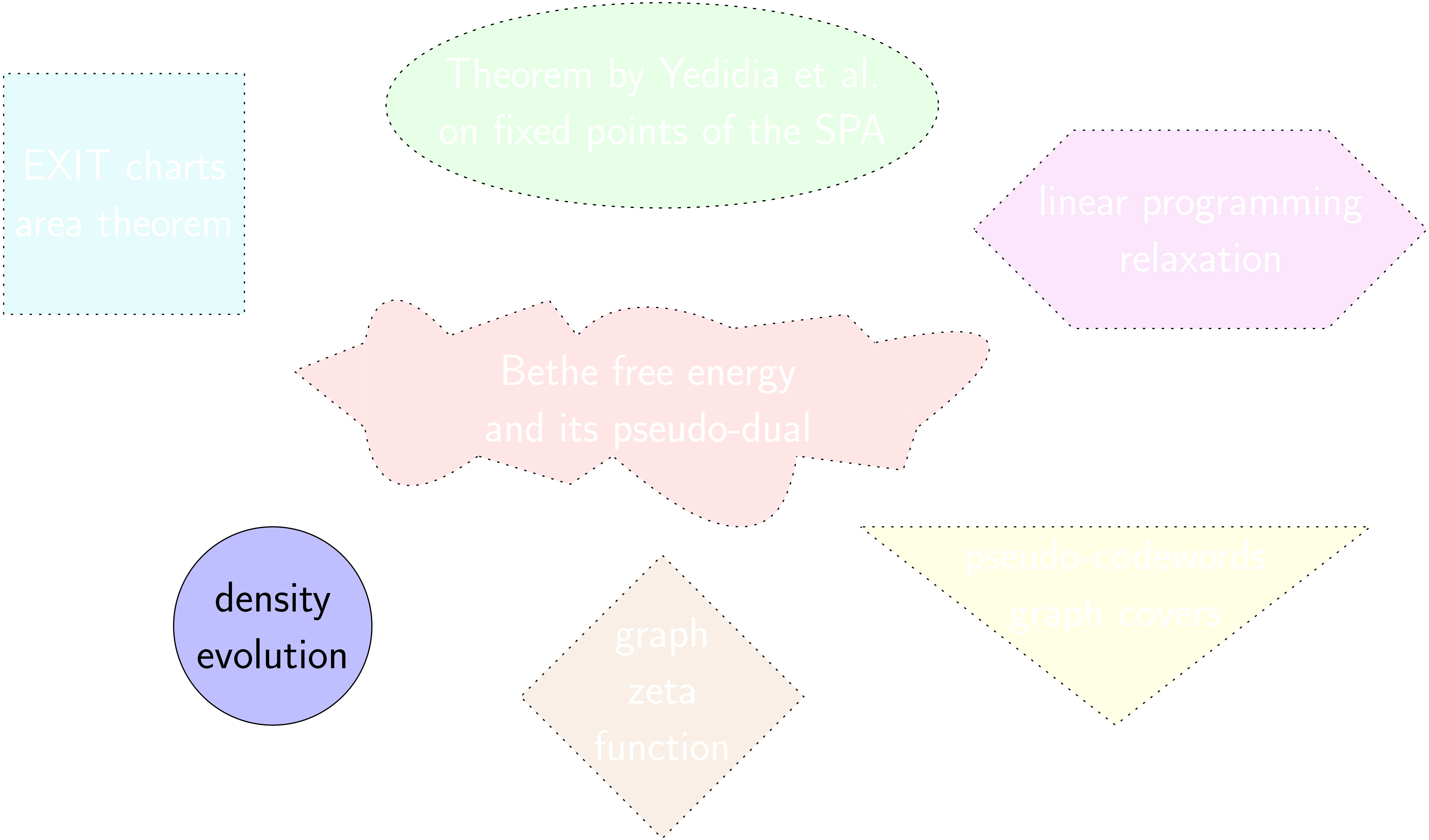

EXIT charts
area theorem
Theorem by Yedidia et al.
on fixed points of the SPA
linear programming
relaxation
Bethe free energy
and its pseudo-dual
density
evolution
graph
zeta
function
pseudo-codewords
graph covers



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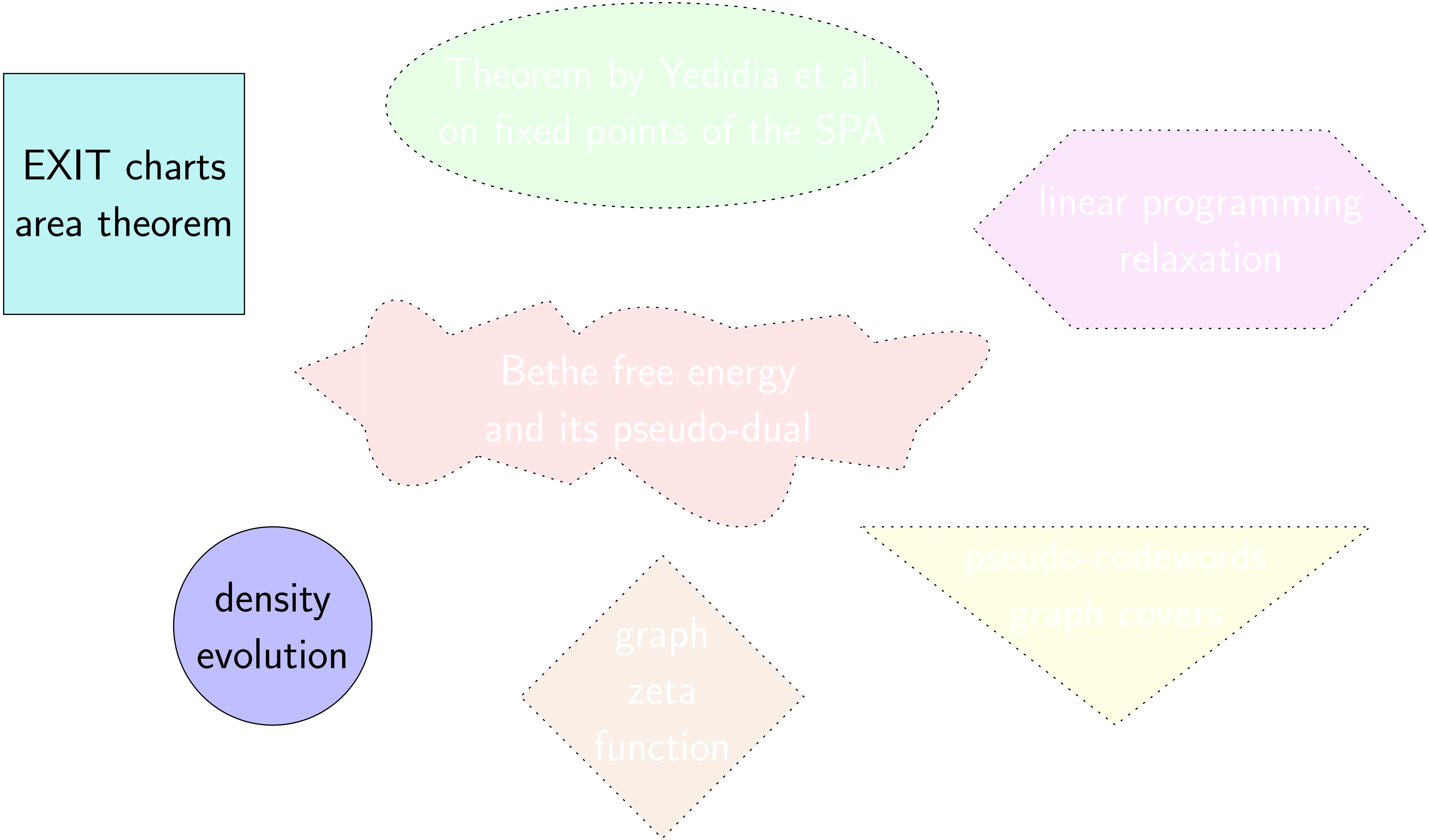

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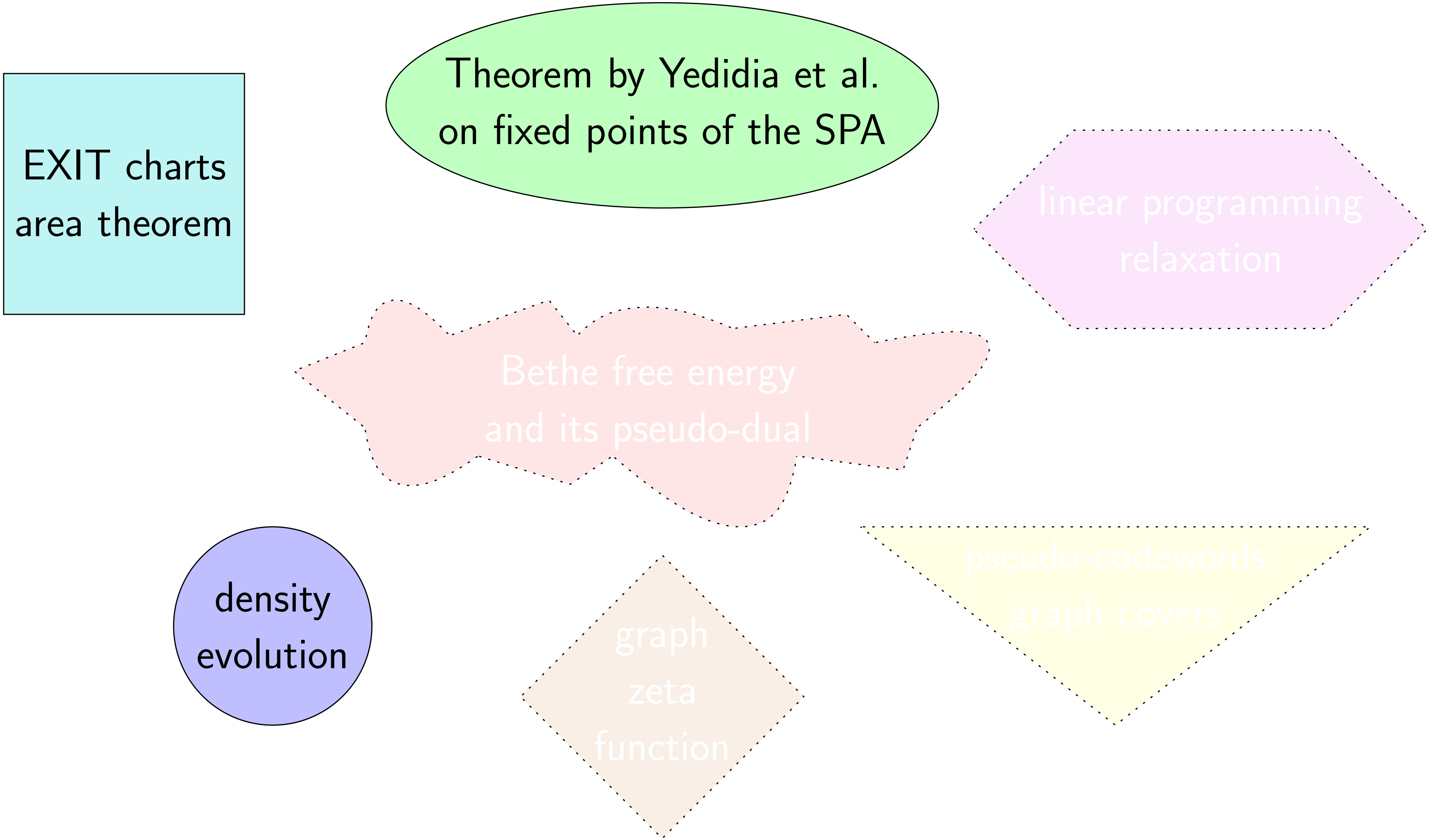

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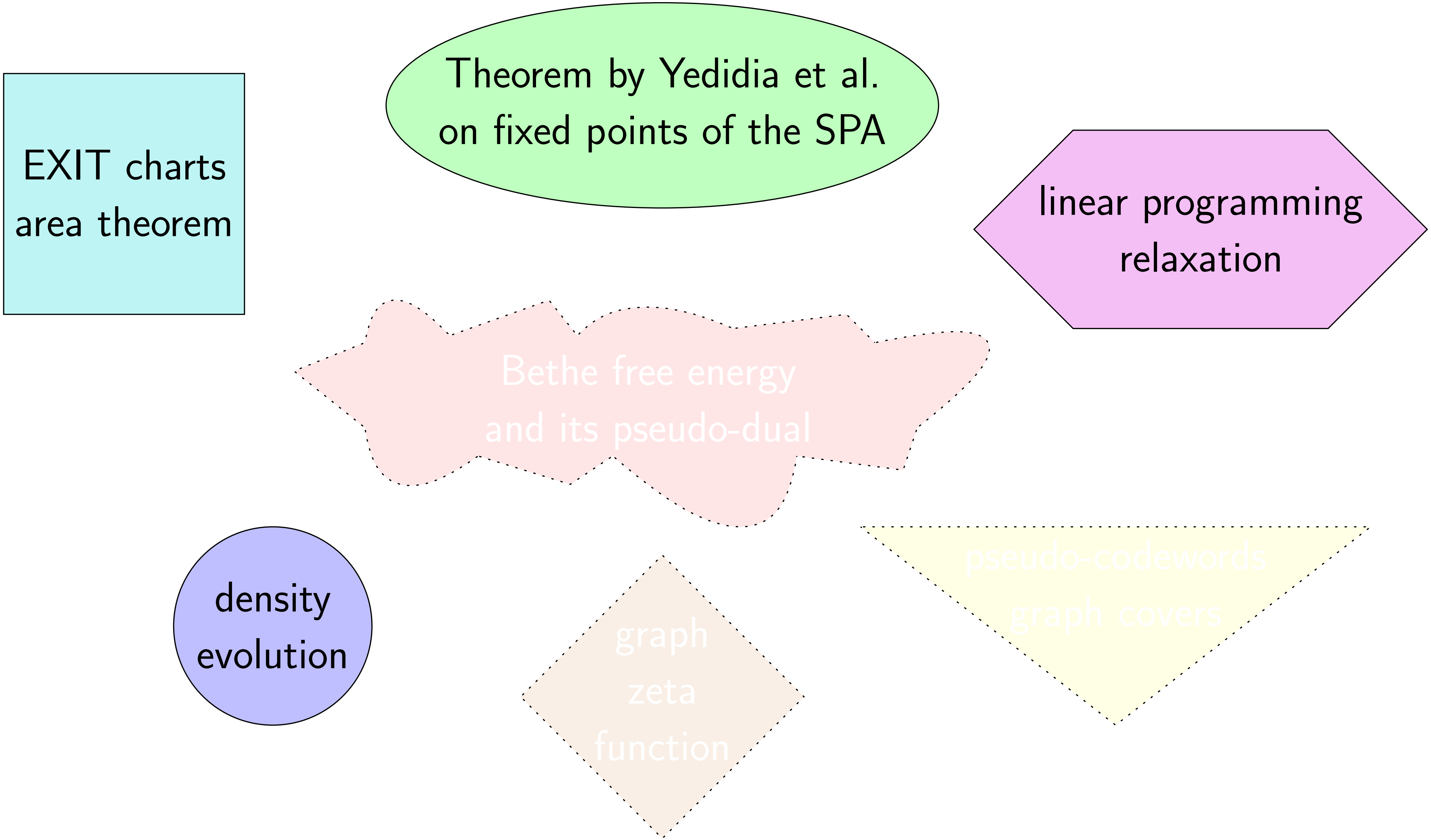

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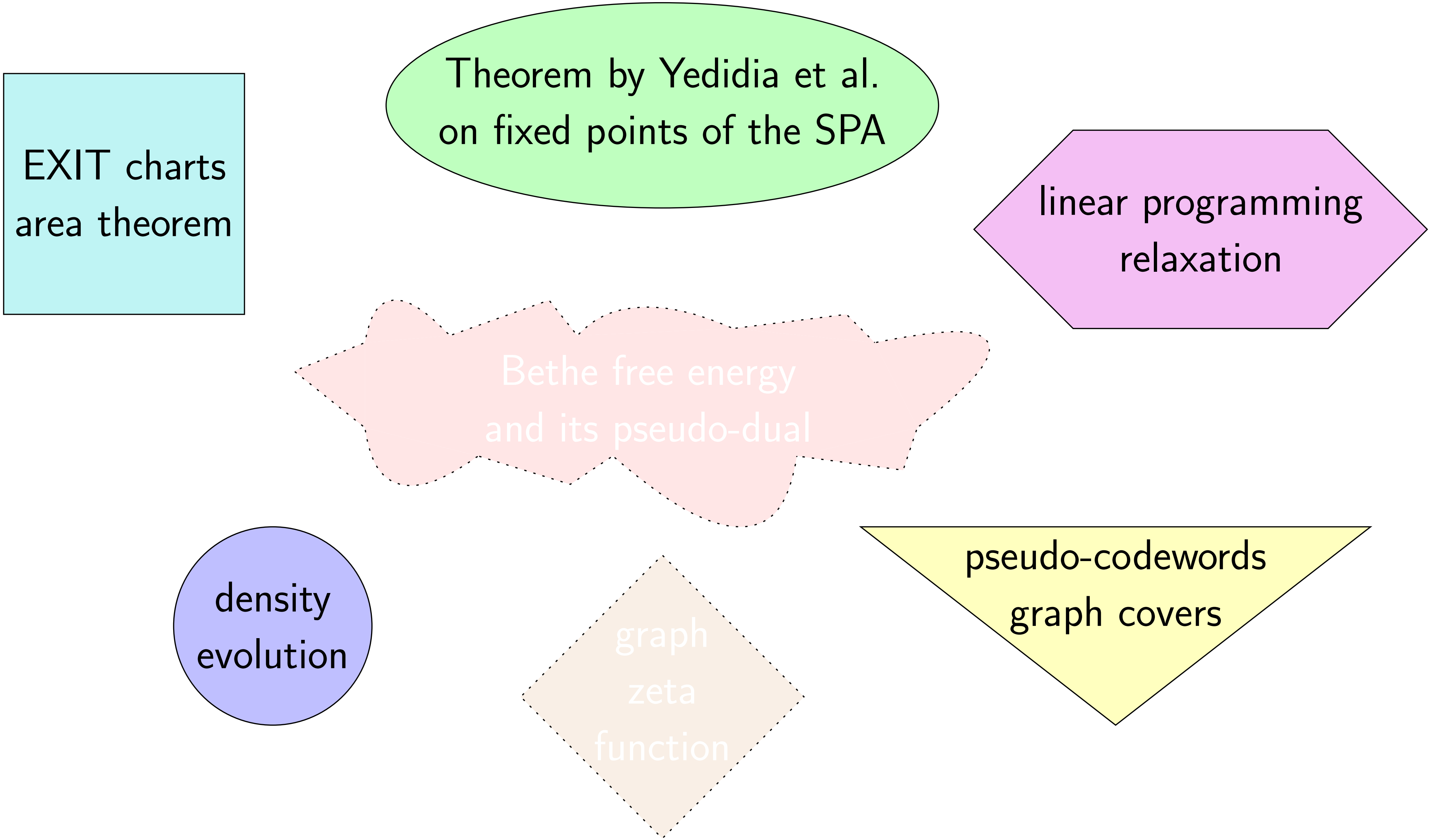

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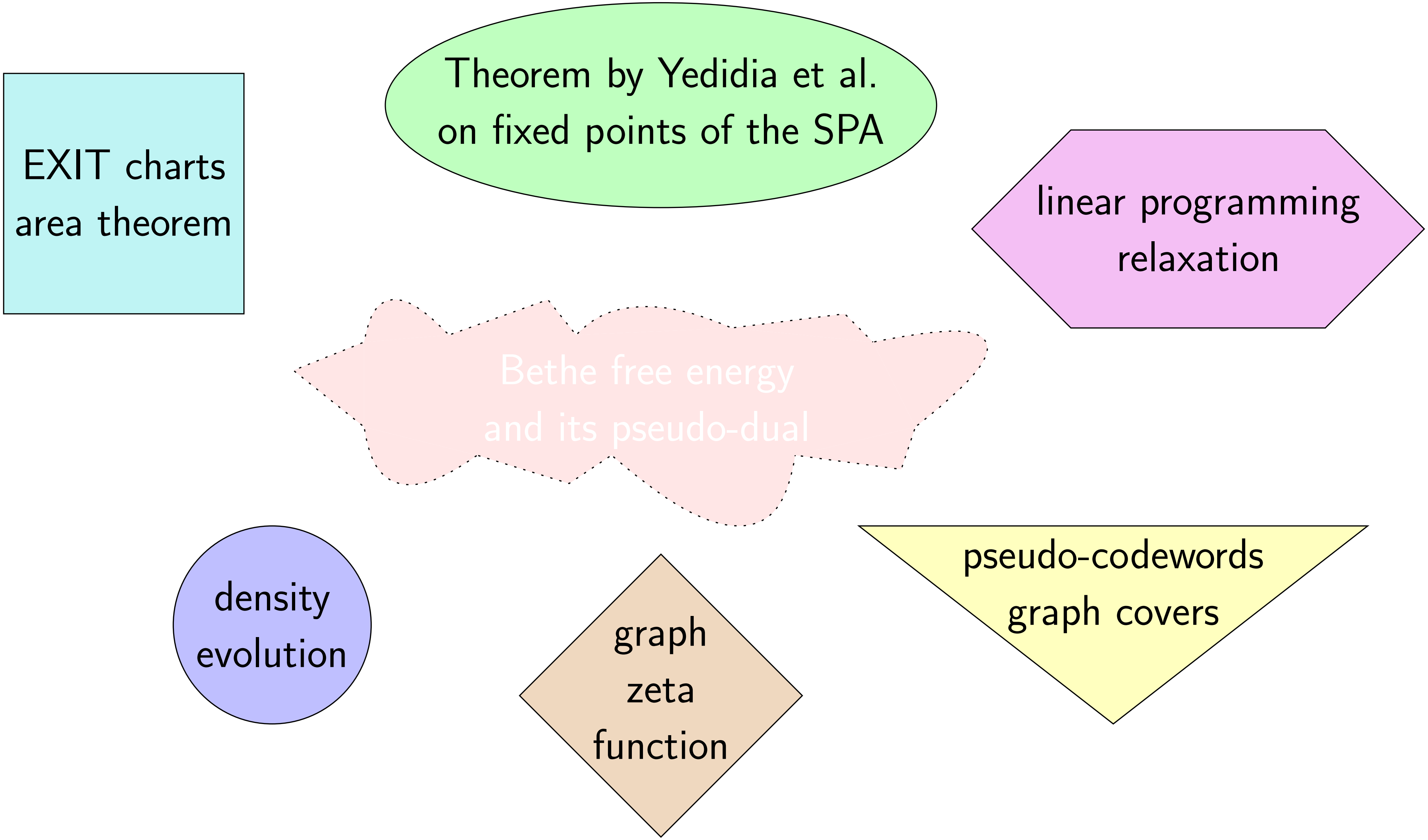

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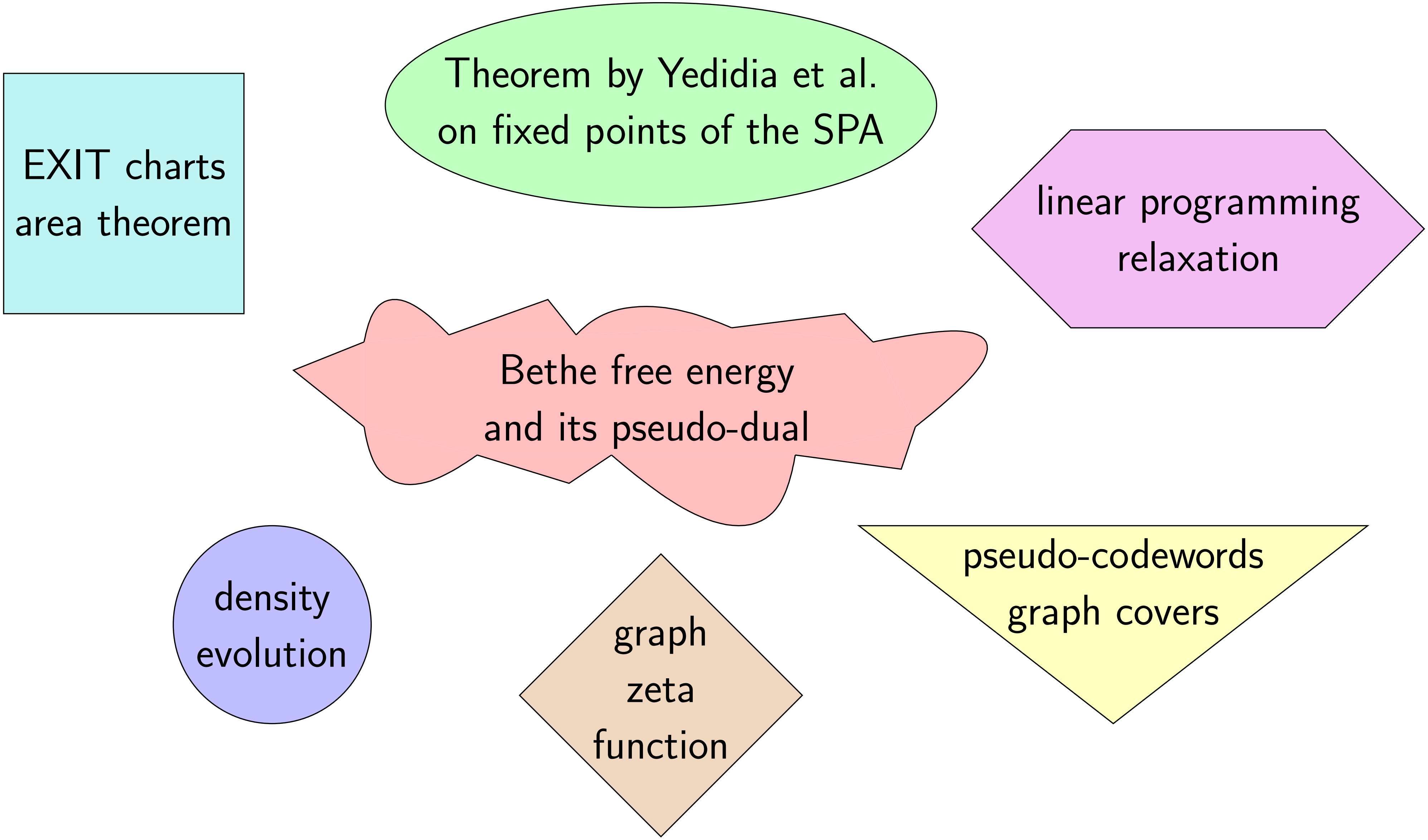

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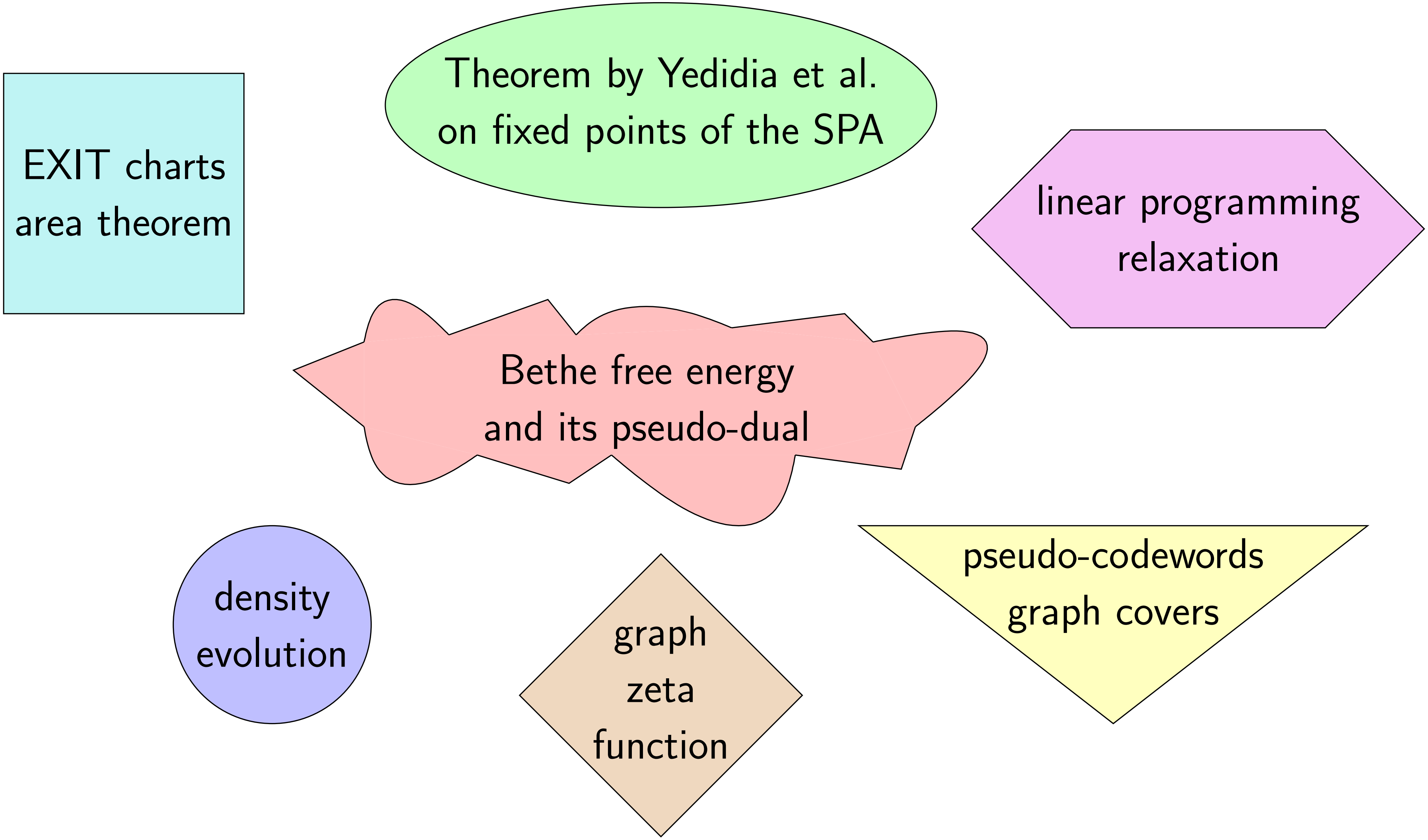

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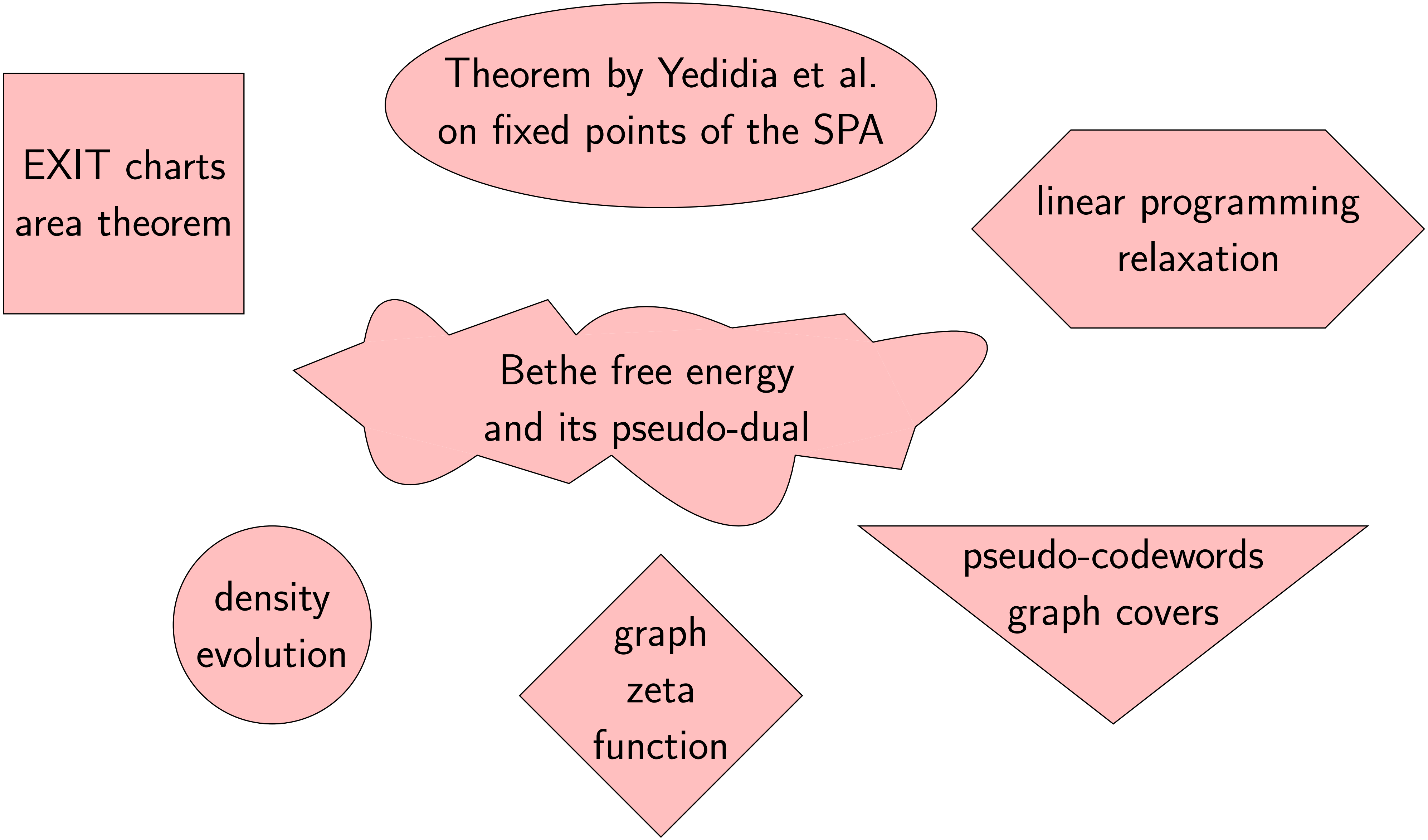

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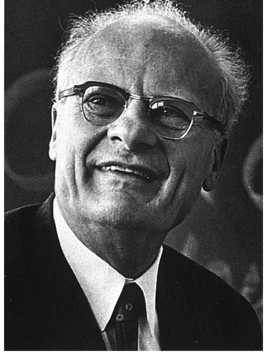

Hans Bethe

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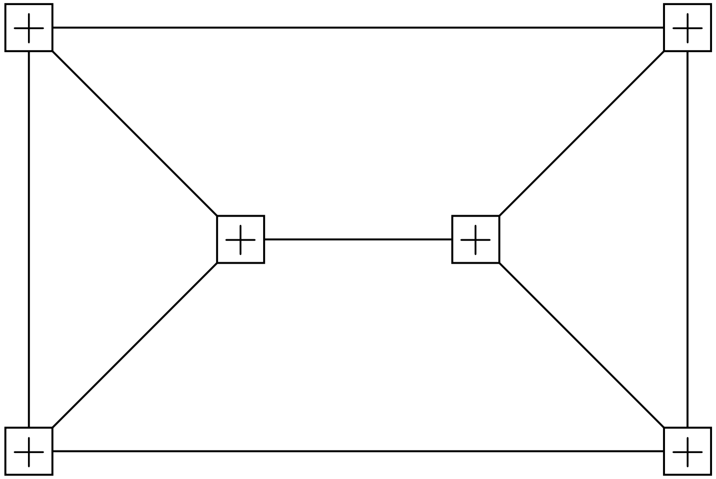
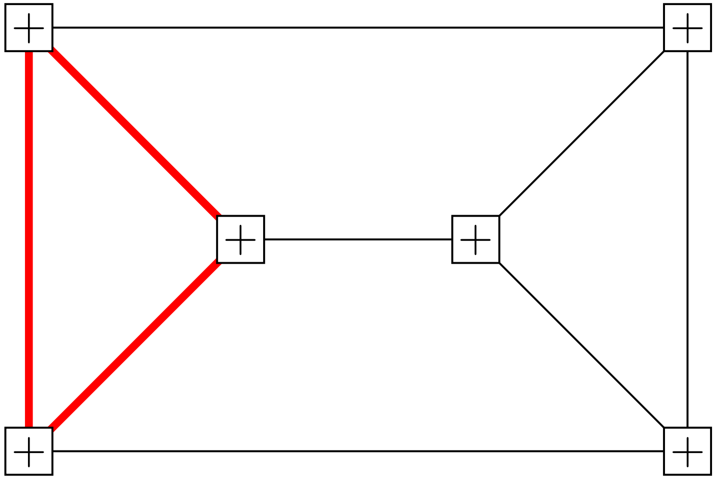
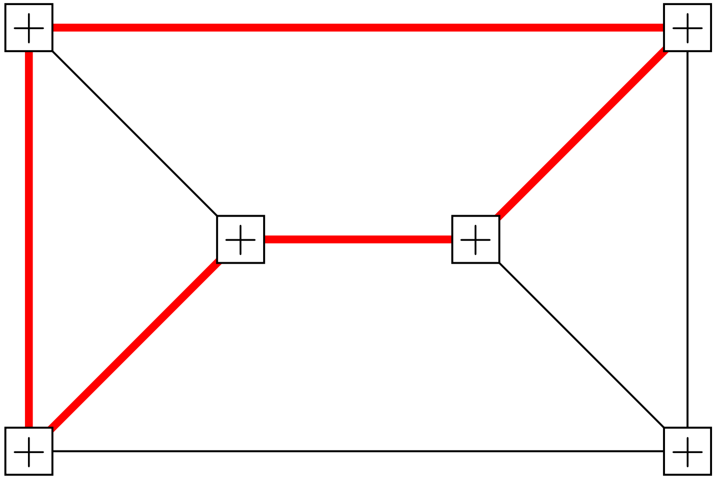
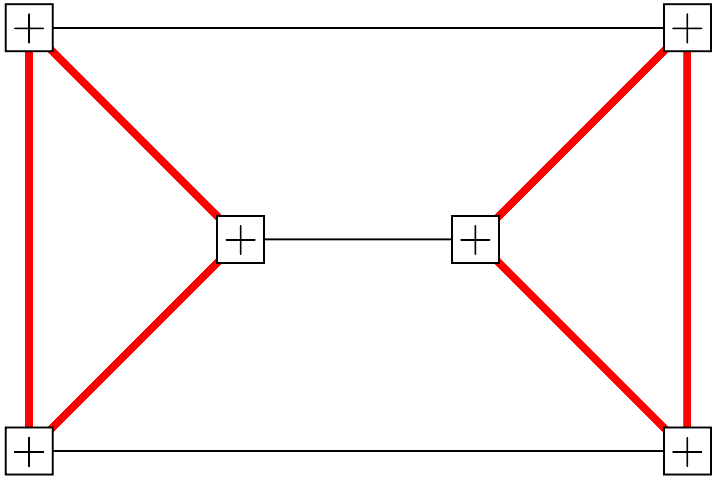
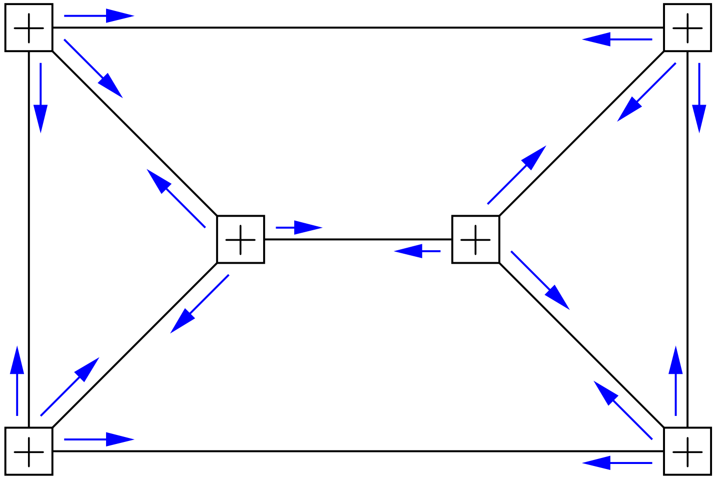
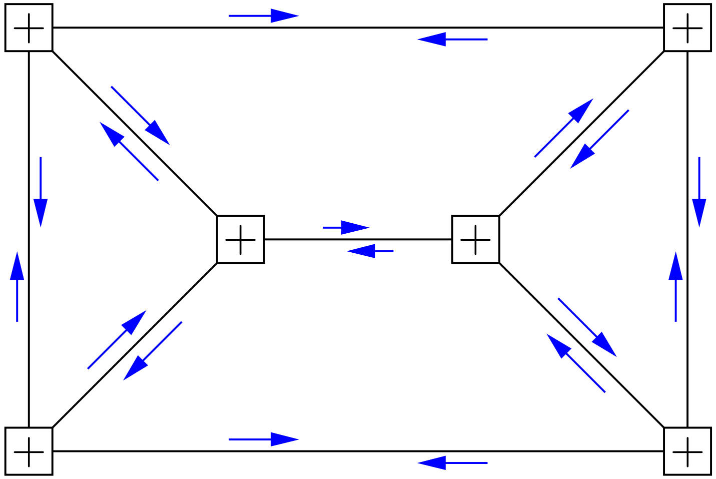


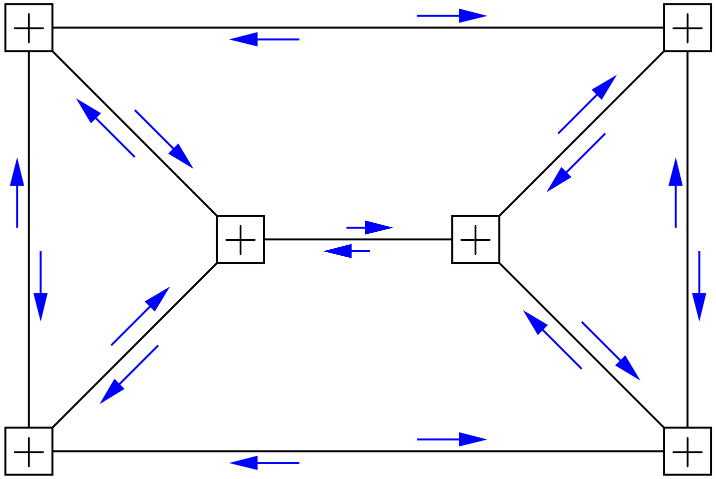


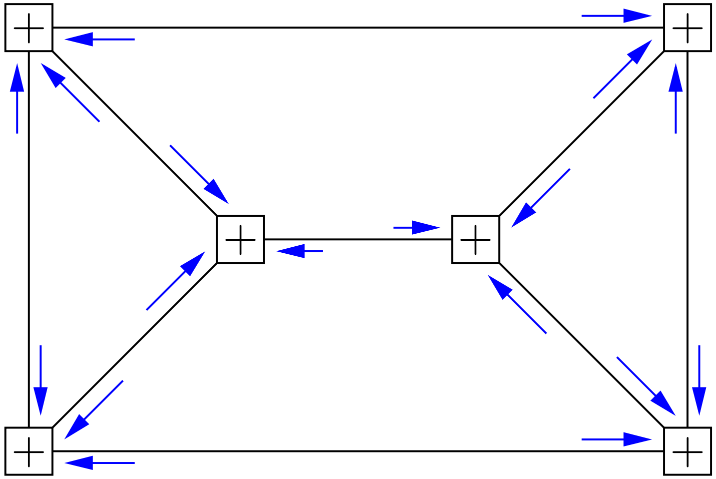


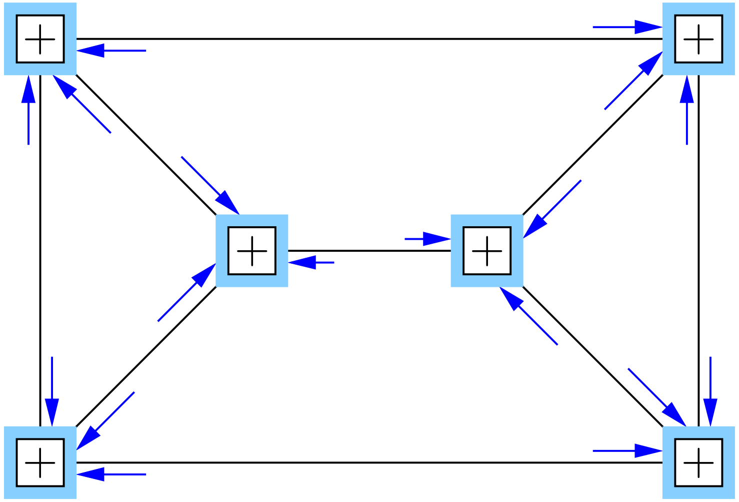


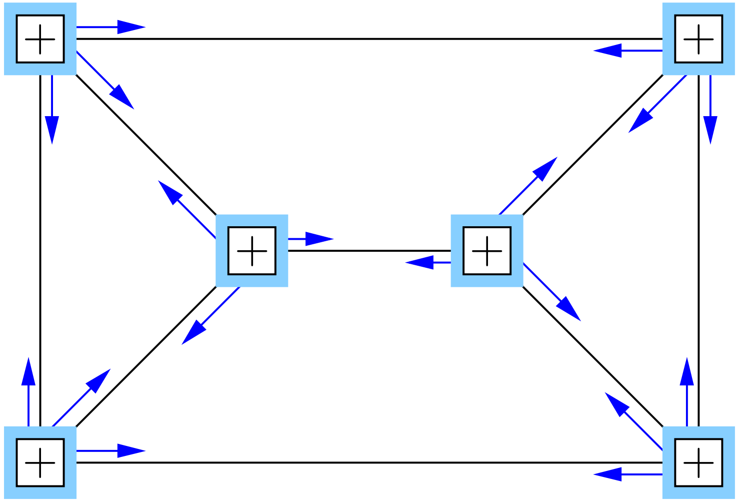


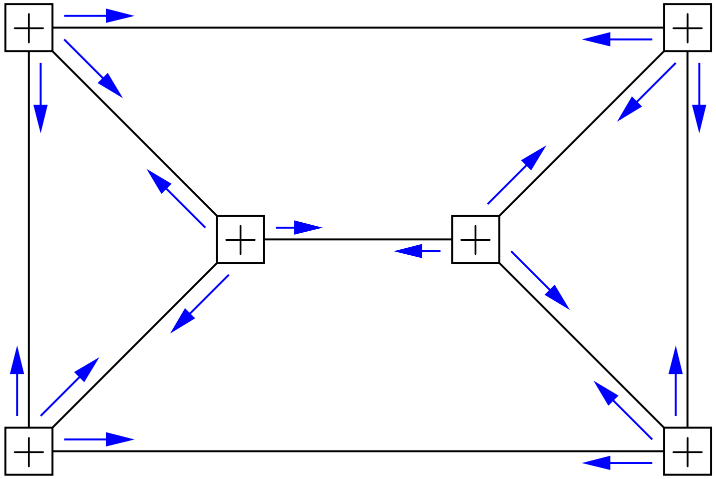


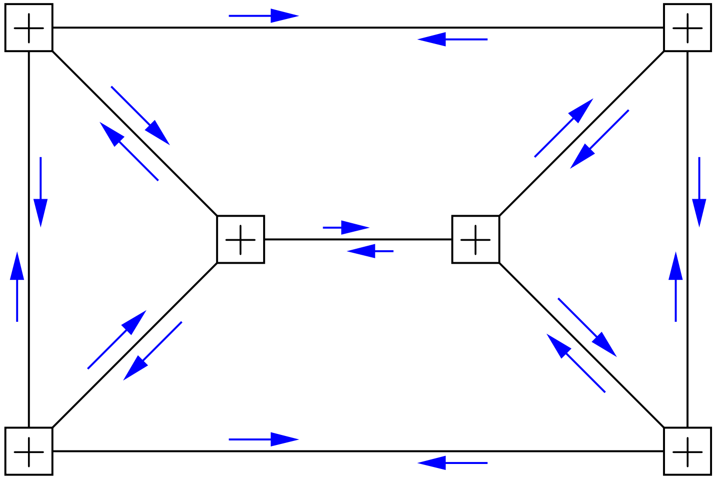


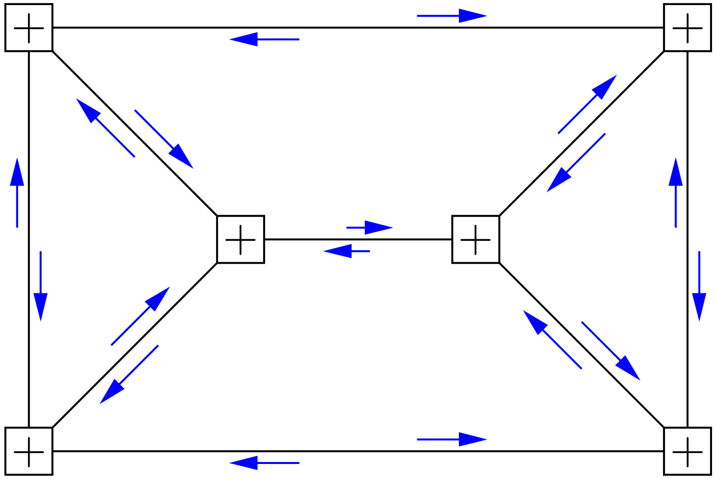


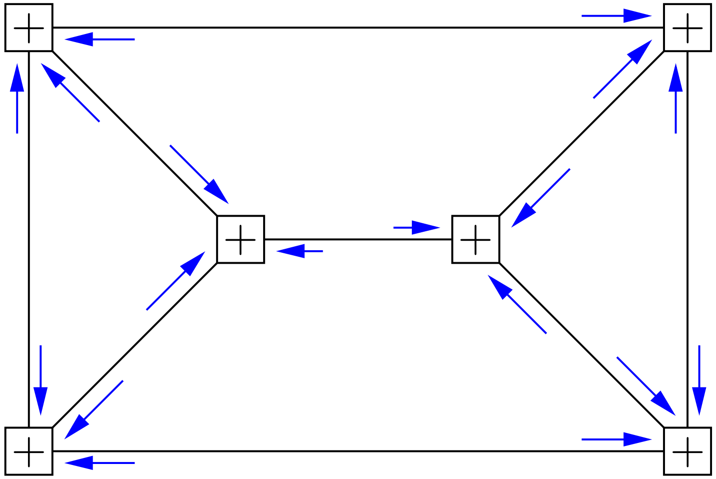


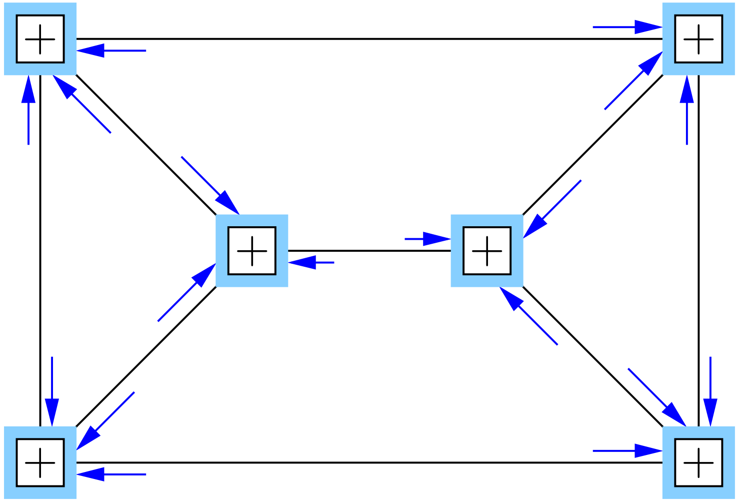


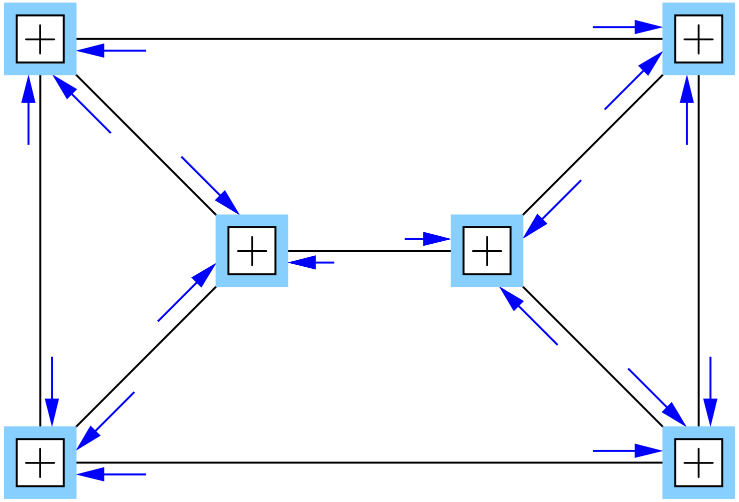


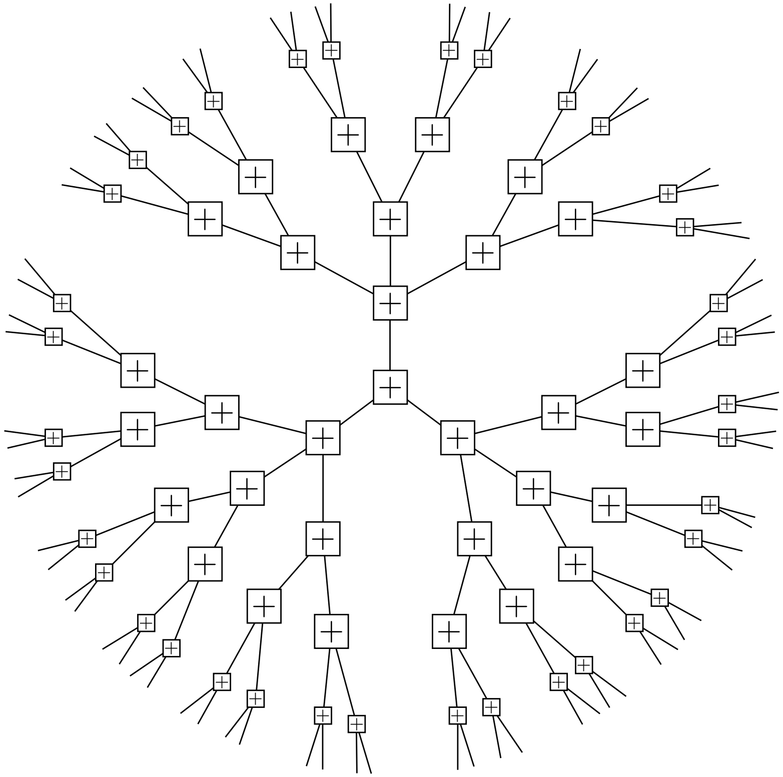


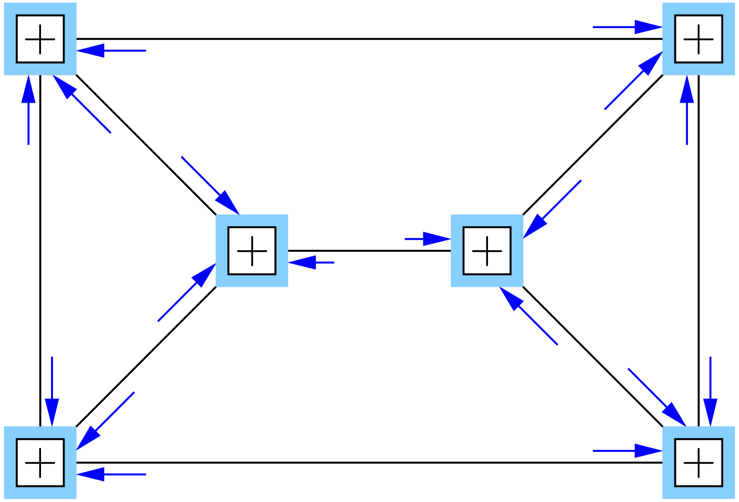


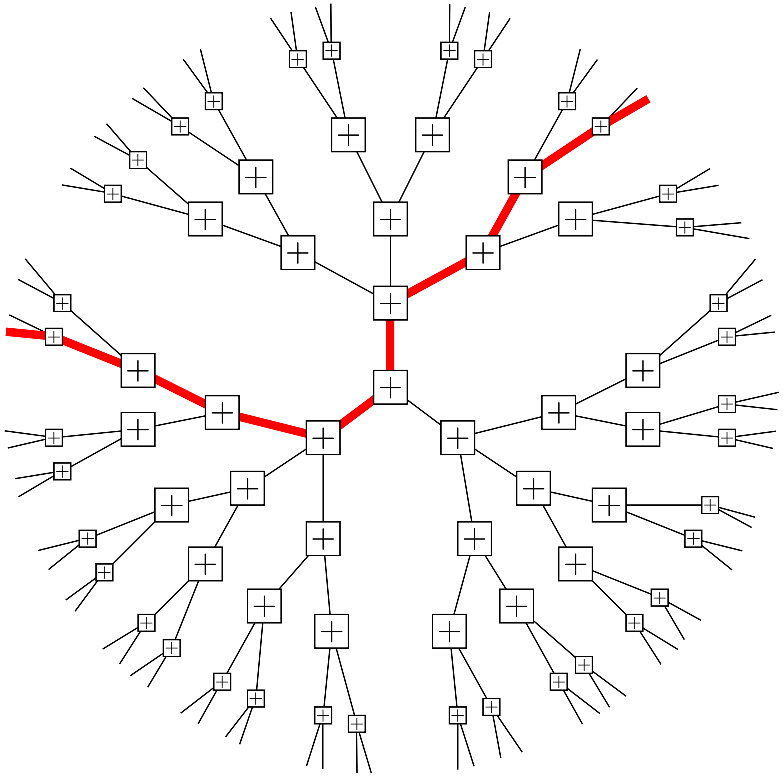


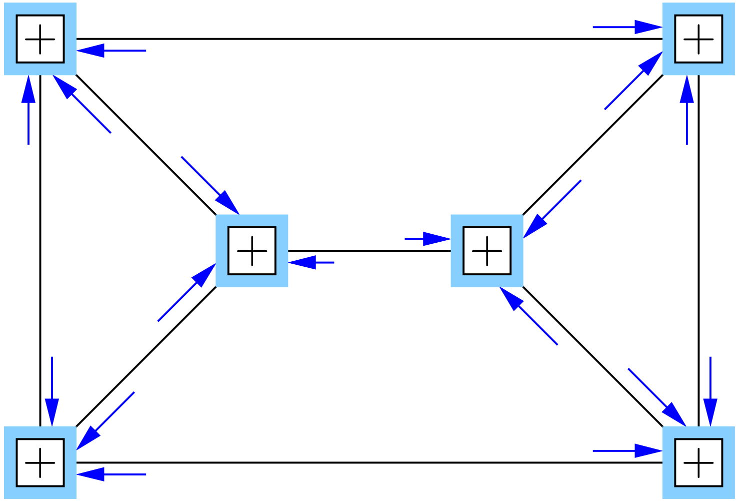


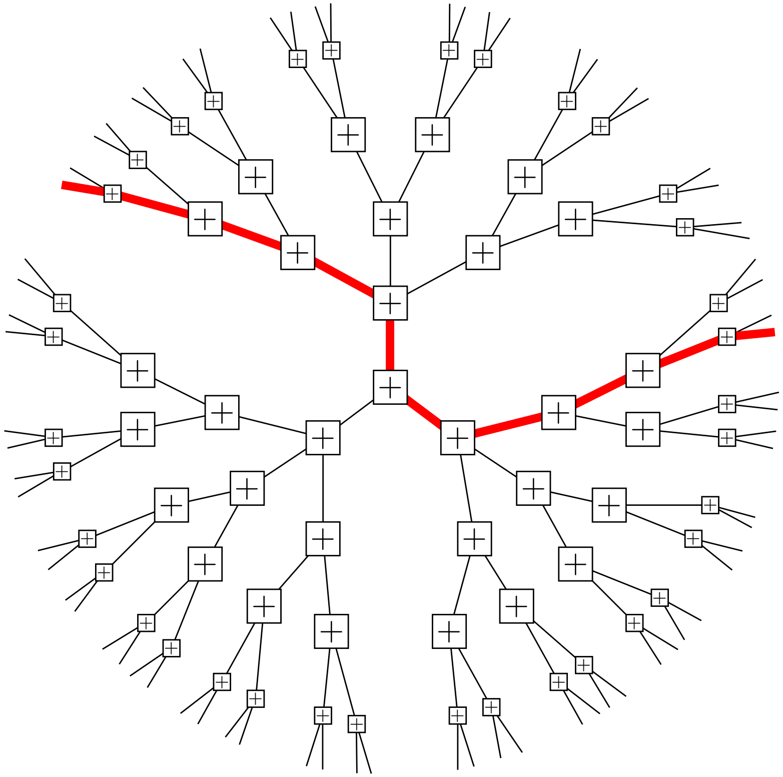


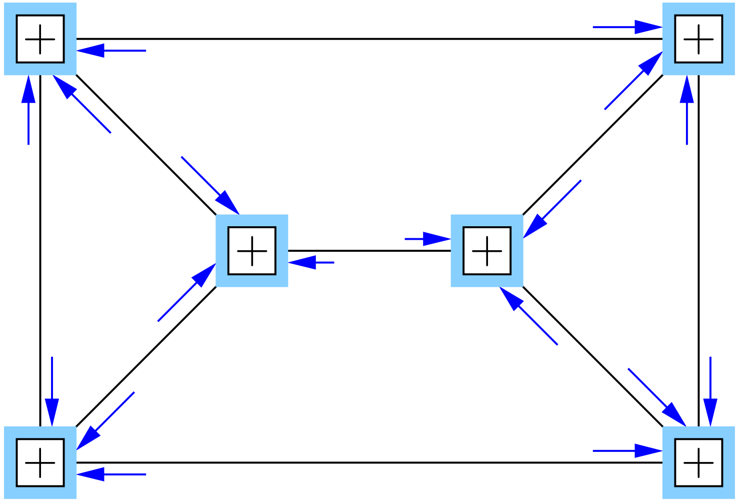


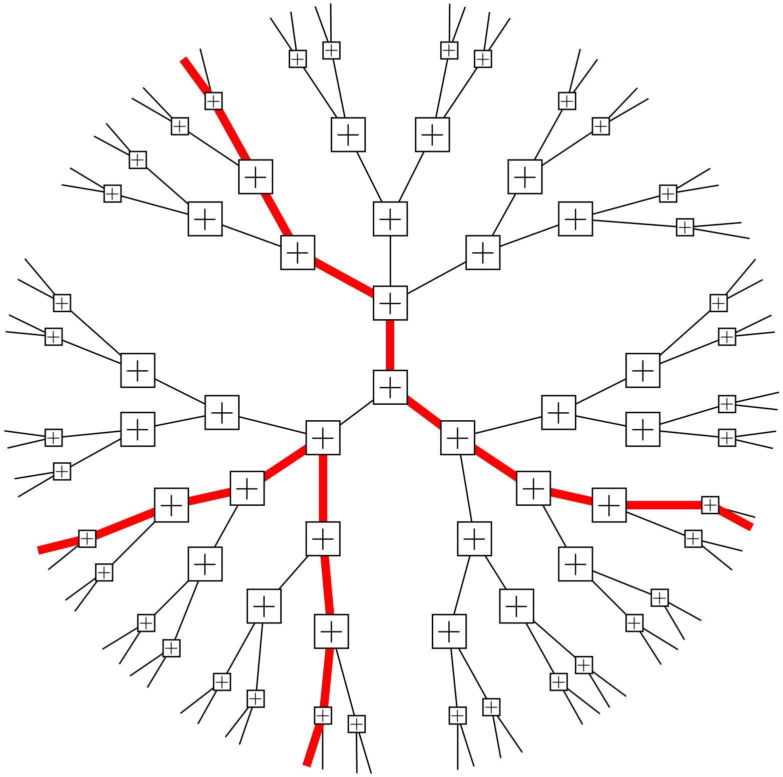


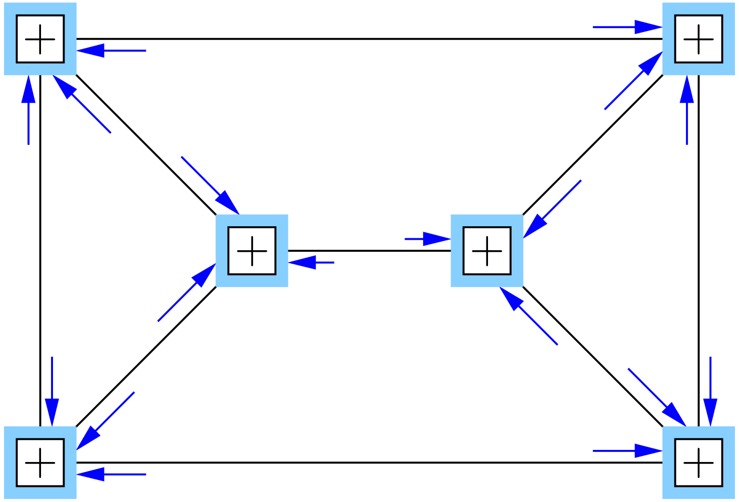


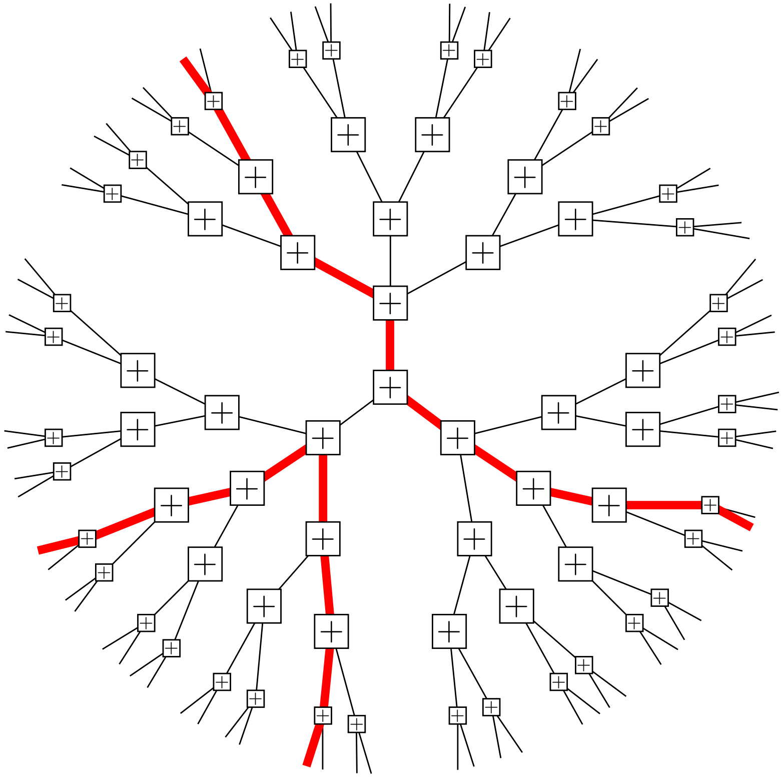


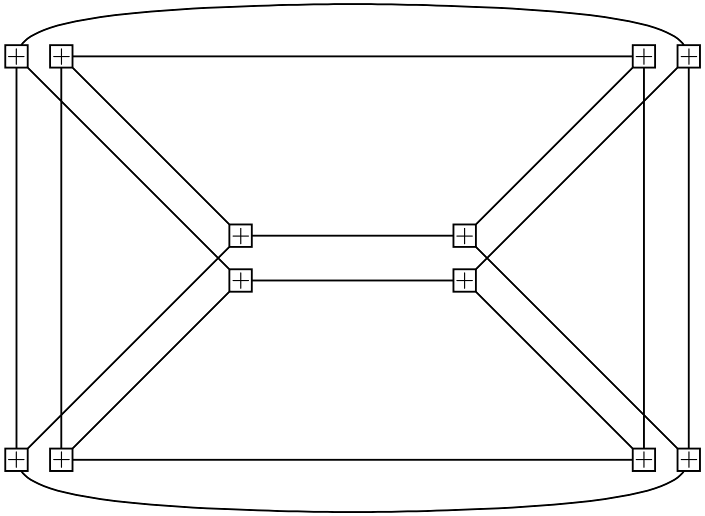


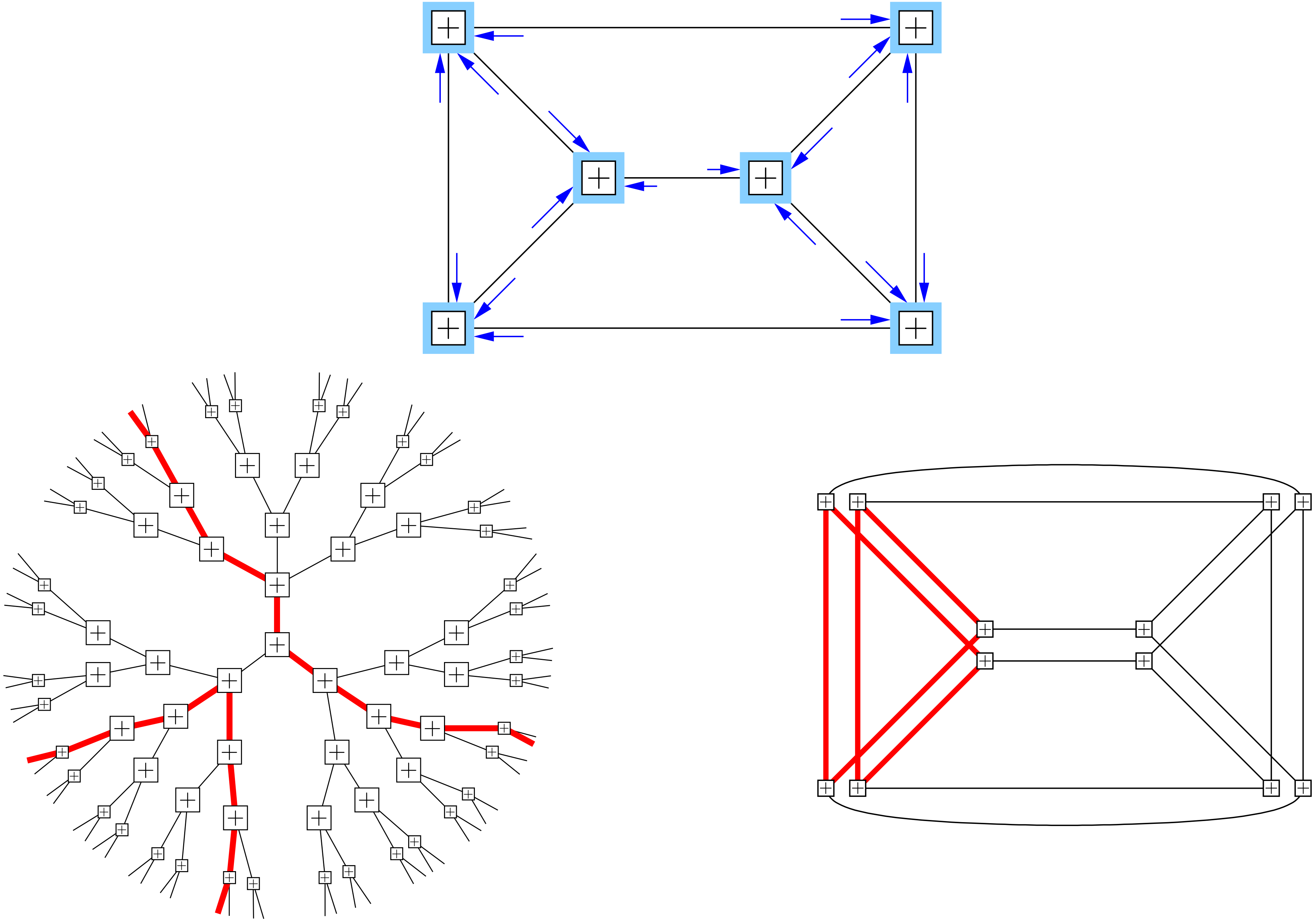


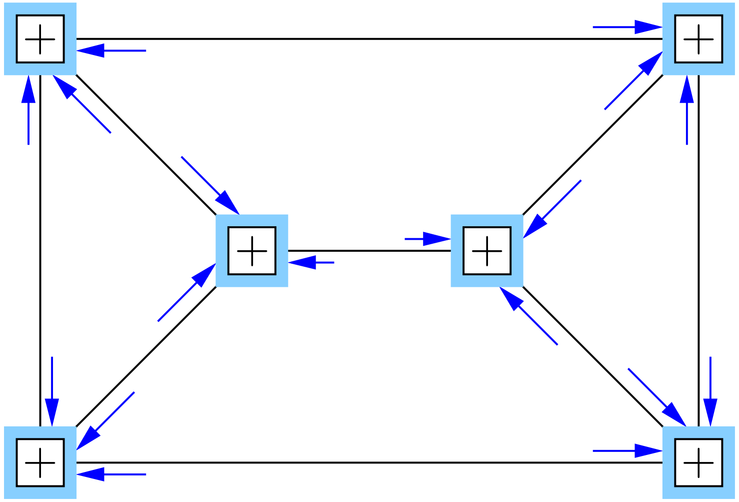


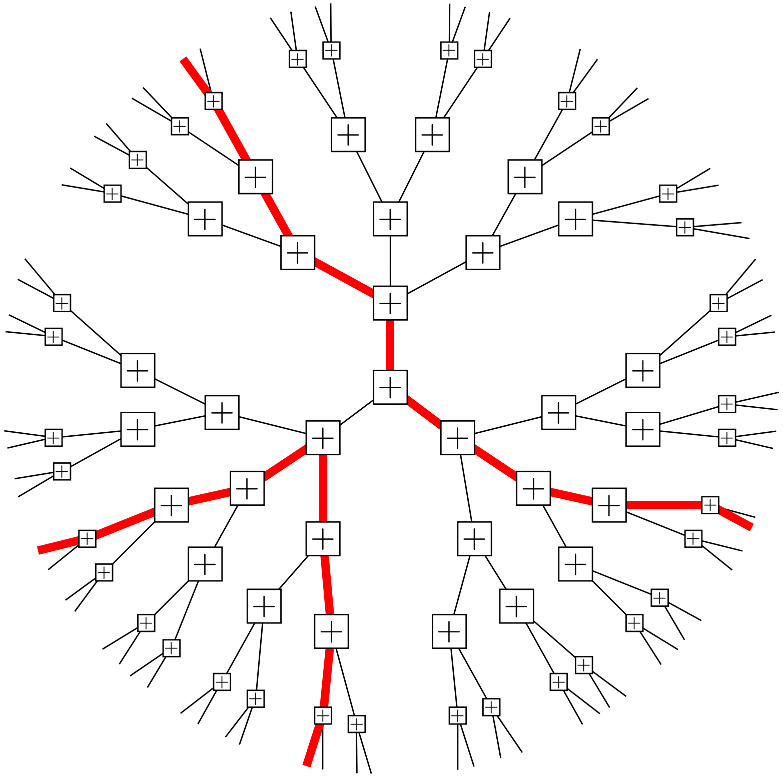


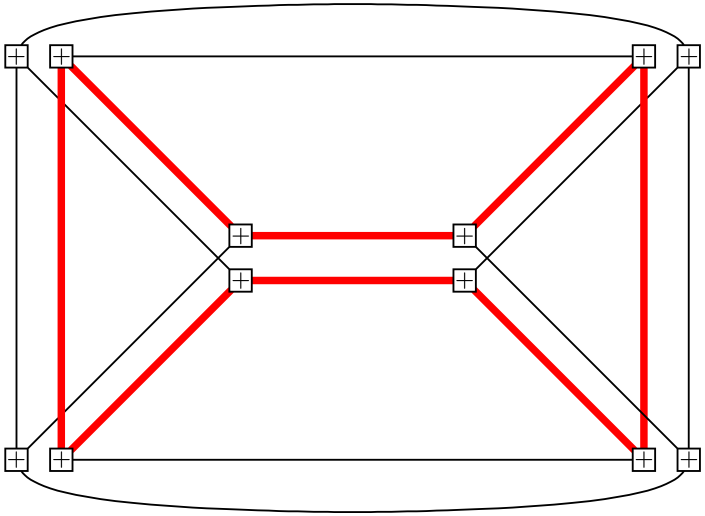


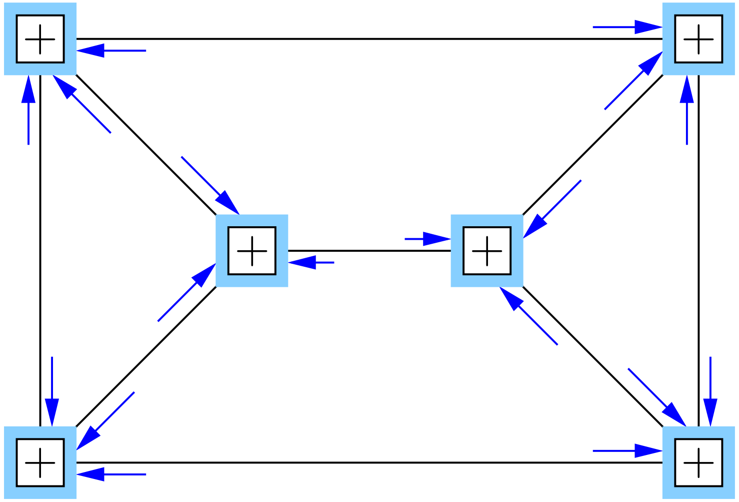


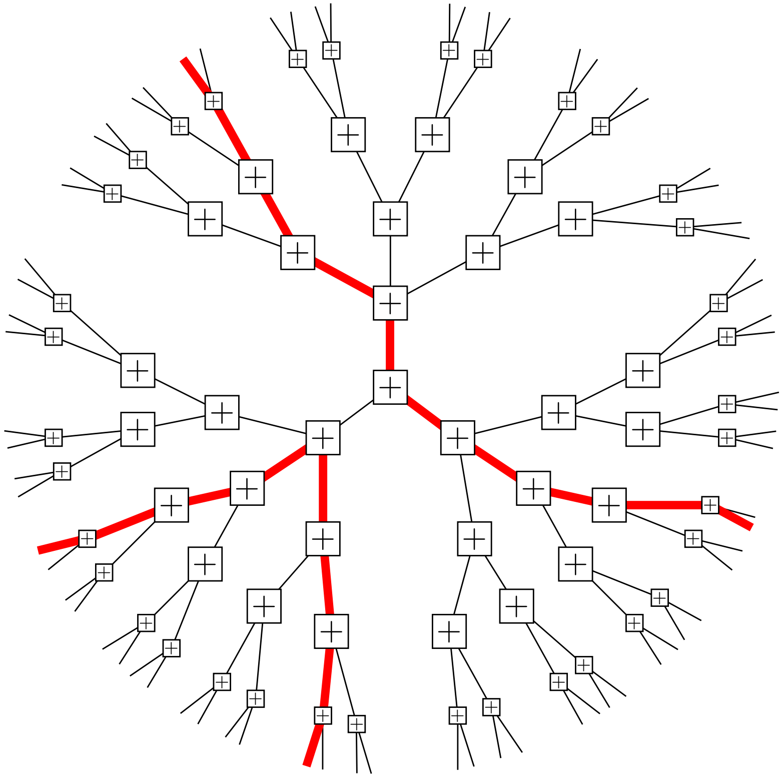


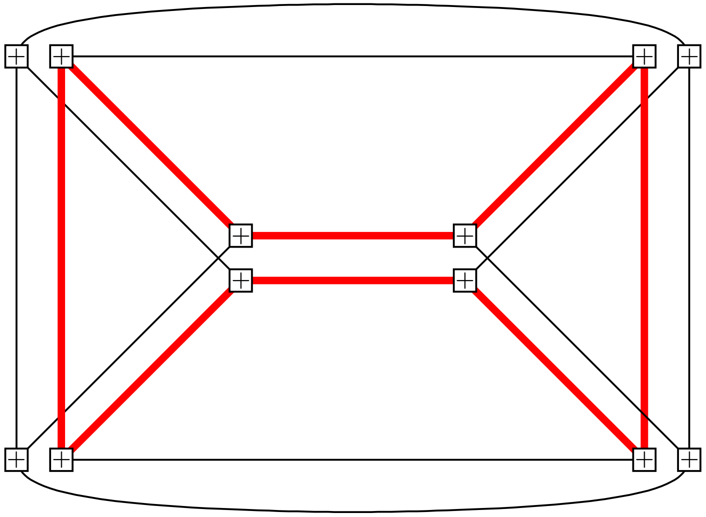




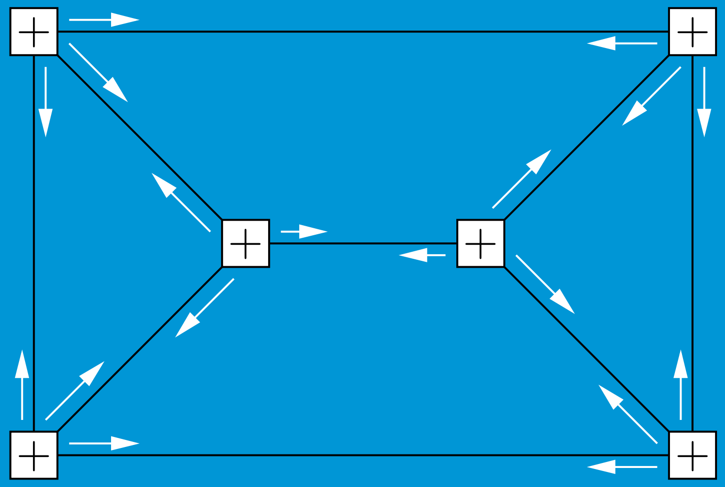


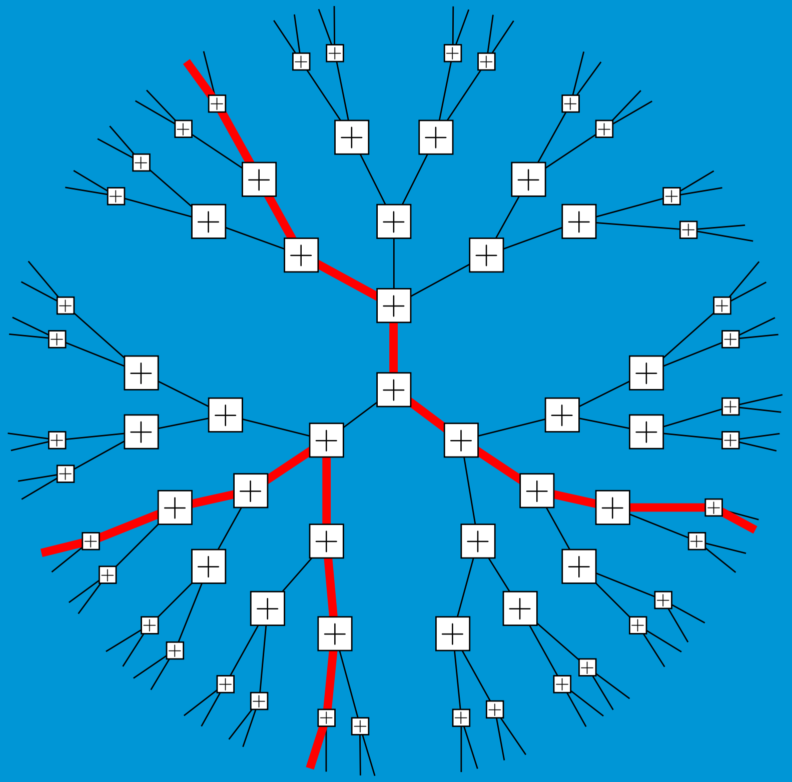


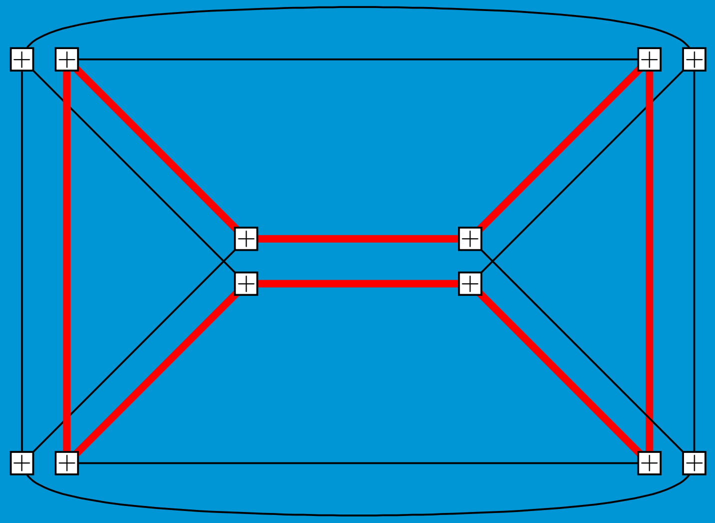




$$\zeta(\mathsf{V}_1,\ldots,\mathsf{V}_n) = \sum_{\mathbf{k}} \zeta_{\mathbf{k}} \mathbf{V}^{\mathbf{k}} = \prod_{[\Gamma]} \frac{1}{1-g(\Gamma,\mathbf{V})}$$







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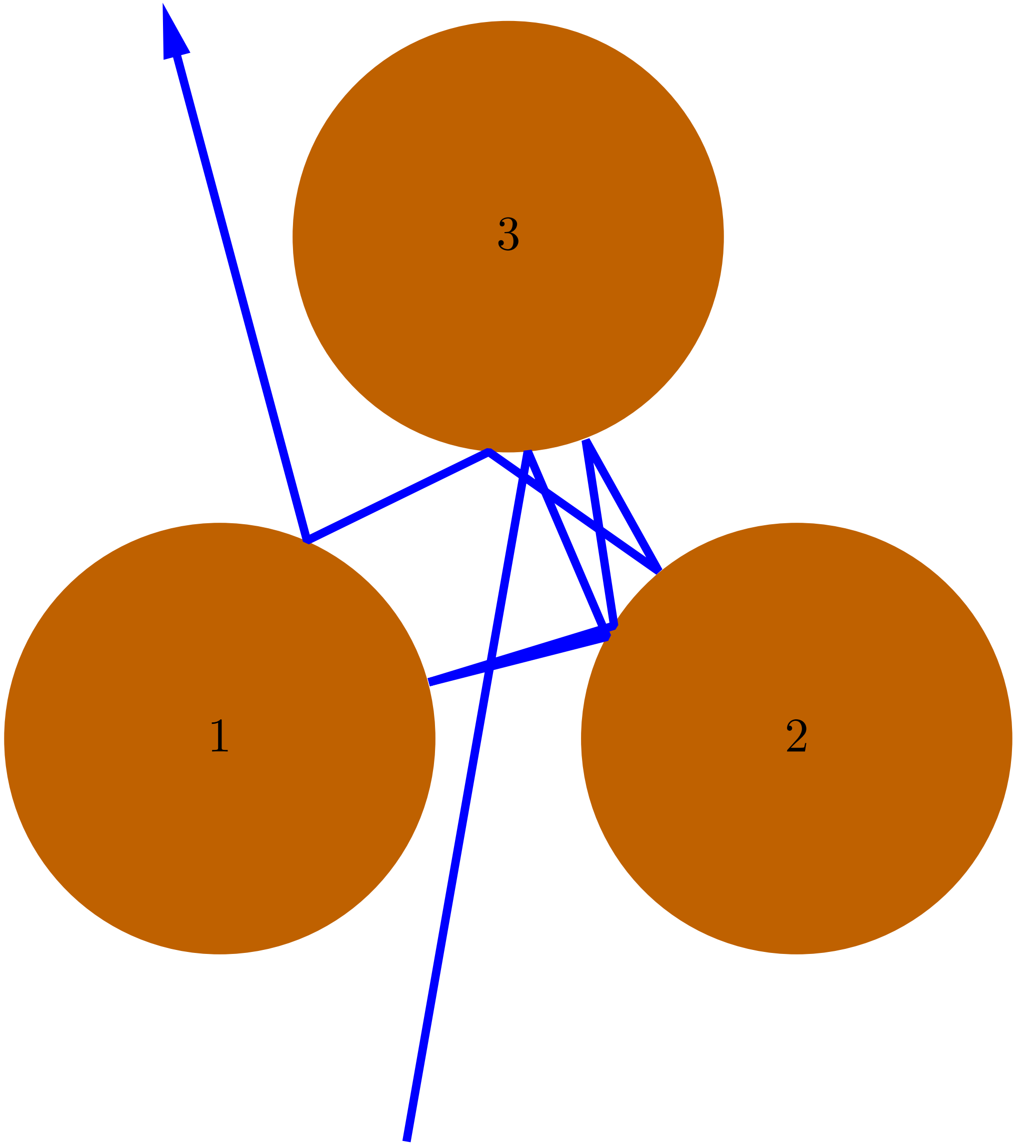
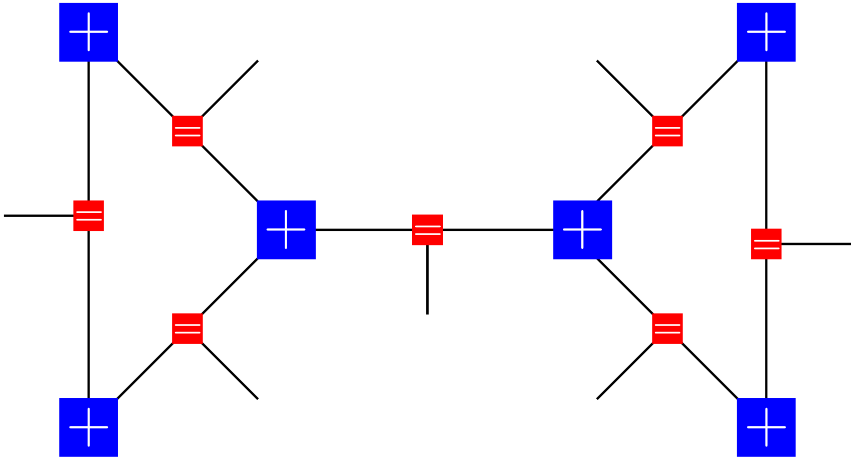
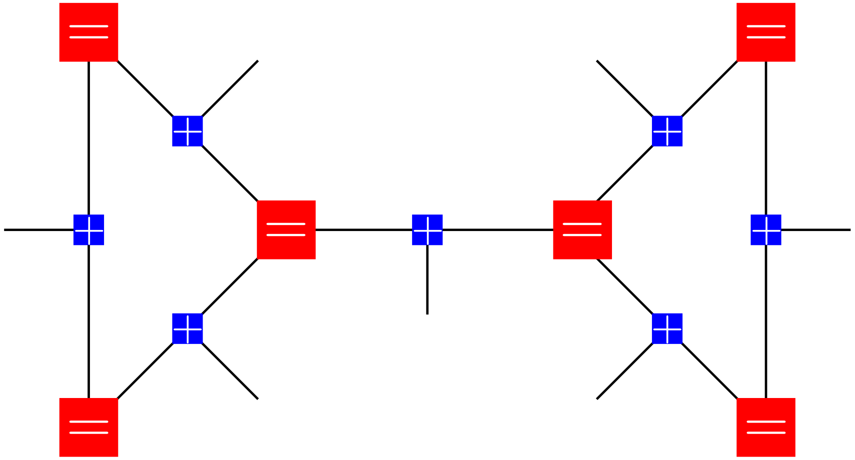
Pinball



Cycle Code NFG –vs– Community Det. NFG



cycle code
normal factor graph



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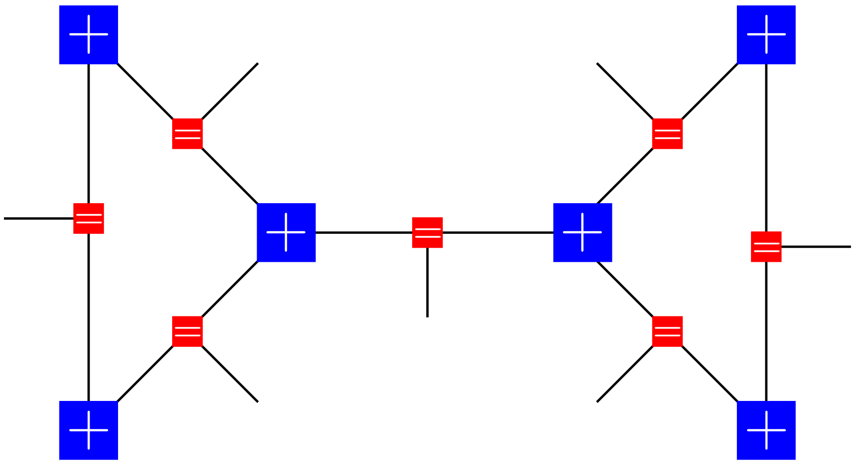

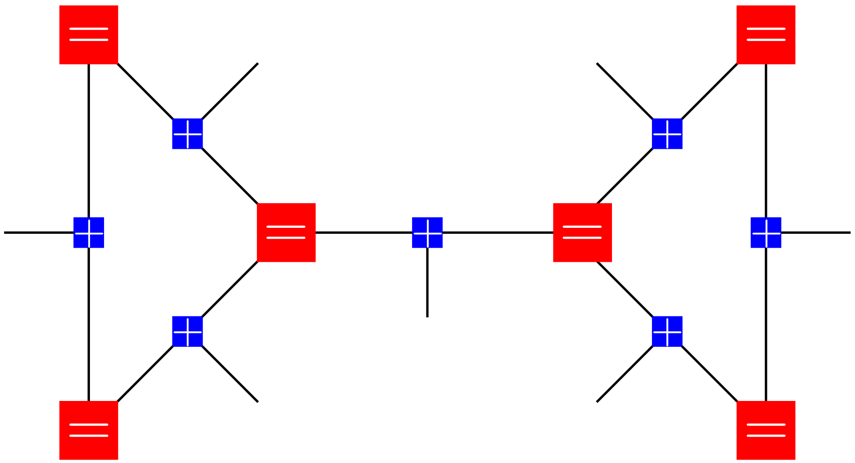

cycle code
normal factor graph

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community detection
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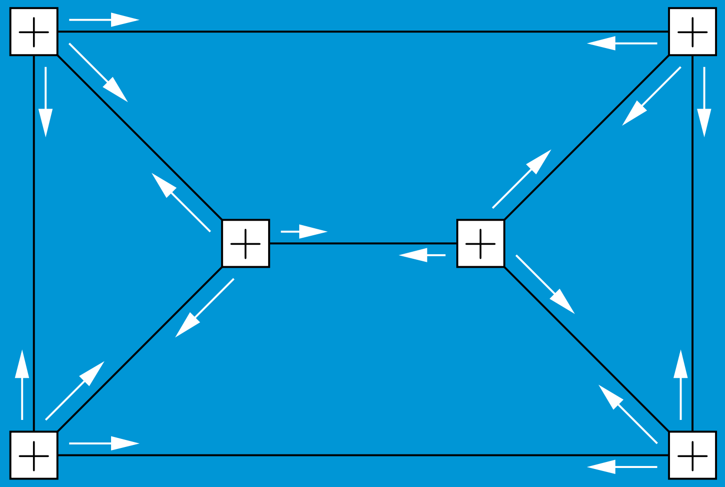
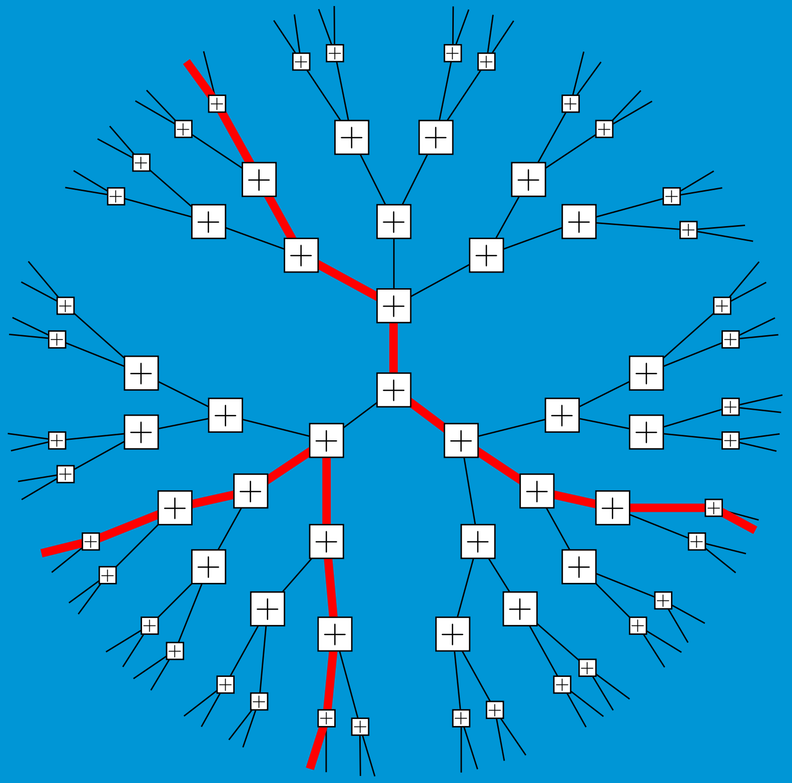
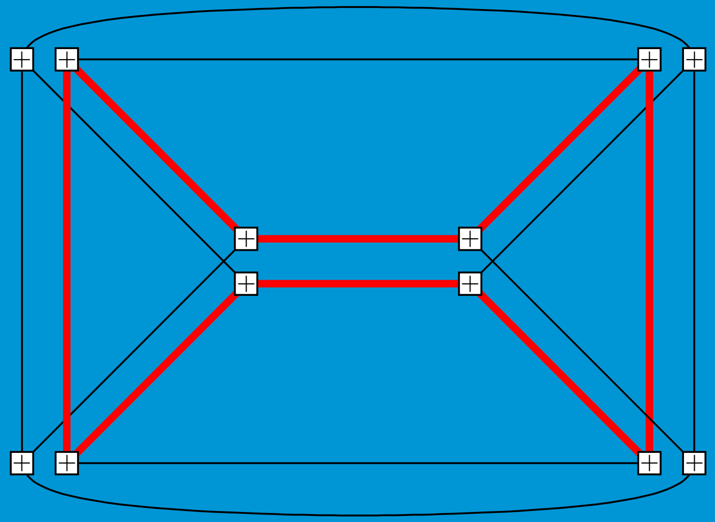
Connection given by normal factor graph duality, cf. [Forney, 2001].







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What Can a Power Series Do For You?

Consider the power series $\theta(\mathsf{V})$:

$$\theta(\mathsf{V}) \triangleq \sum_{k} \theta_k \mathsf{V}^k = 1\mathsf{V}^0 + 2\mathsf{V}^1 + 8\mathsf{V}^3 + 16\mathsf{V}^4 + 64\mathsf{V}^6 + 128\mathsf{V}^7 + \cdots$$

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- . . .

What Can a Power Series Do For You?

Consider the power series $\theta(\mathsf{V}_1, \ldots, \mathsf{V}_n)$:

$$\theta(\mathsf{V}_1, \ldots, \mathsf{V}_n) \triangleq \sum_{k_1,\ldots,k_n} \theta_{k_1,\ldots,k_n} \mathsf{V}_1^{k_1} \cdots \mathsf{V}_n^{k_n}$$

We can obtain useful information from

- ...the exponent vectors of $\theta(\mathsf{V}_1, \ldots, \mathsf{V}_n)$
- ...the coefficients of $\theta(\mathsf{V}_1, \ldots, \mathsf{V}_n)$
- ...the evaluation of $\theta(\mathsf{V}_1, \ldots, \mathsf{V}_n)$ for some $(\mathsf{V}_1, \ldots, \mathsf{V}_n)$
- ...the convergence region of $\theta(\mathsf{V}_1, \ldots, \mathsf{V}_n)$
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What Can a Power Series Do For You?

Consider the power series (zeta function) $\zeta(\mathbf{V})$:

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Use of zeta functions for analyzing graphical models.

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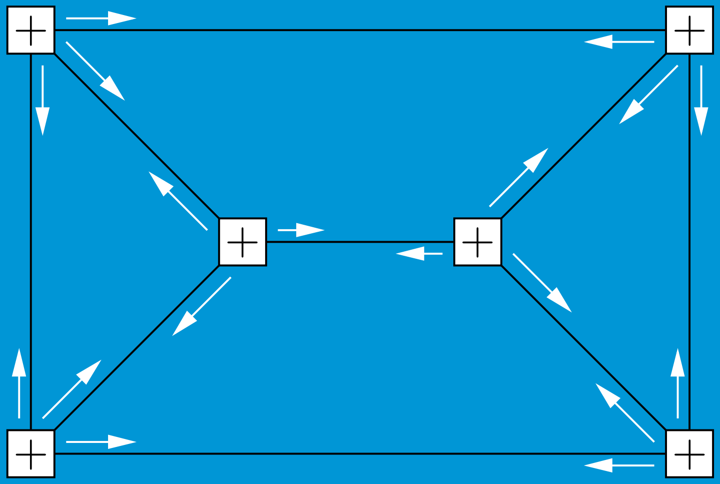
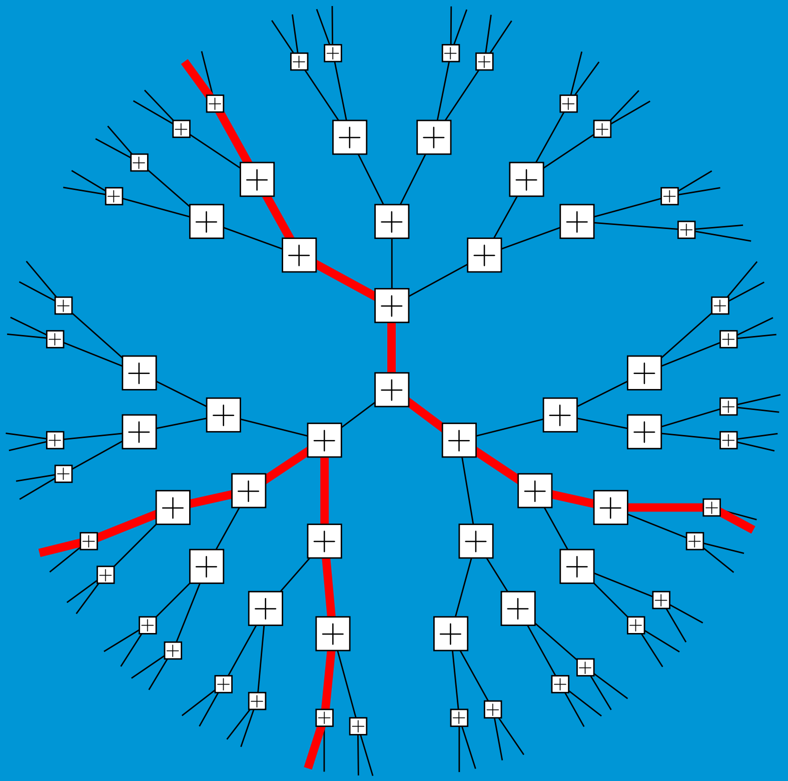
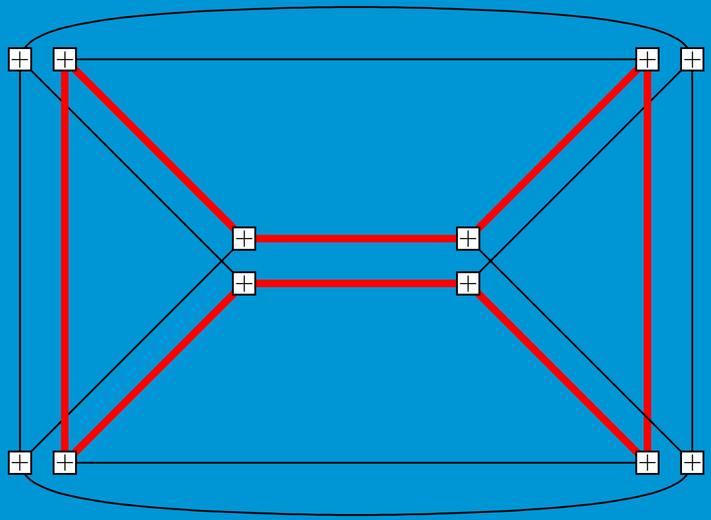
Use of zeta functions for analyzing graphical models.

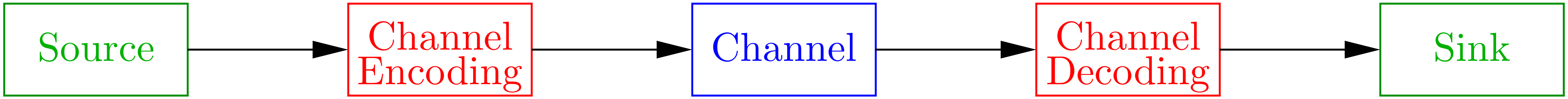






$$\zeta(\mathsf{V}_1,\ldots,\mathsf{V}_n) = \sum_{\mathbf{k}} \zeta_{\mathbf{k}} \mathsf{\mathbf{V}}^{\mathbf{k}} = \prod_{[\Gamma]} \frac{1}{1-g(\Gamma,\mathsf{\mathbf{V}})}$$

Communication Model



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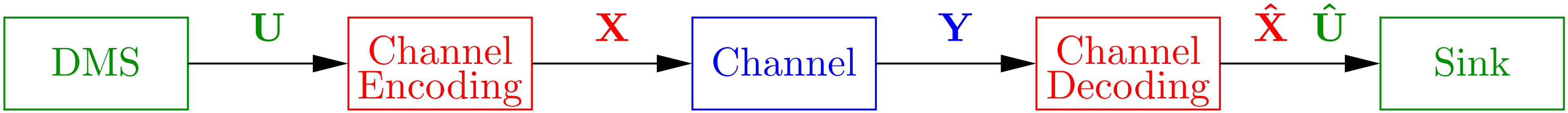

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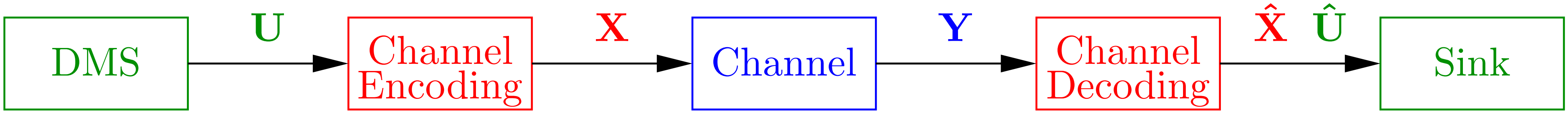

Information word: $\mathbf{u} = (u_1, \ldots, u_k) \in \mathcal{U}^k$

Sent codeword: $\mathbf{x} = (x_1, \ldots, x_n) \in \mathcal{C} \subseteq \mathcal{X}^n$

Received word: $\mathbf{y} = (y_1, \ldots, y_n) \in \mathcal{Y}^n$

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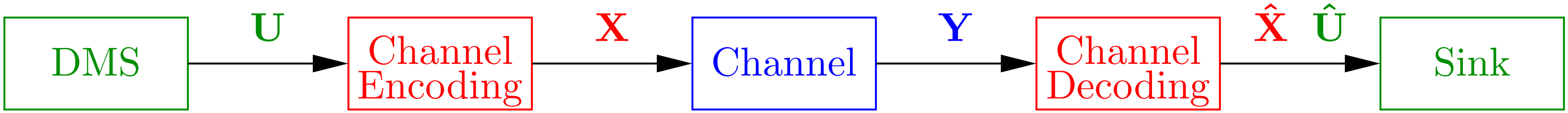

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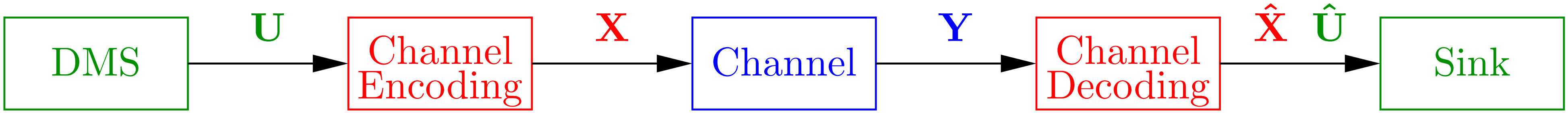

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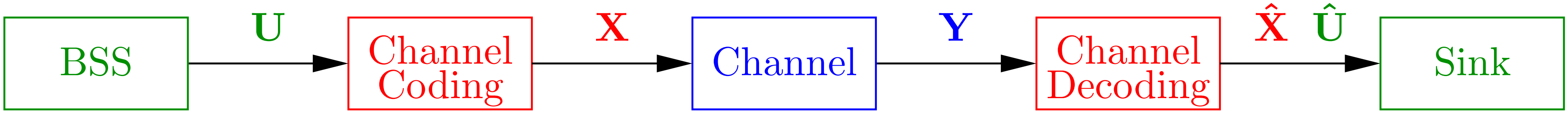
Depending on what criterion we optimize, we obtain different decoding algorithms.

Symbol-Wise MAP Decoding (Part 1)



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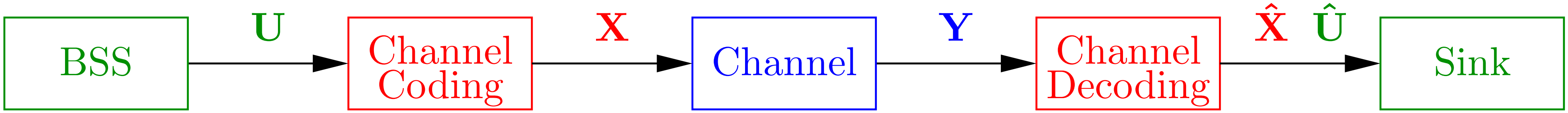

Minimizing the symbol error probability (for each $i = 1, \ldots, k$) results in symbol-wise MAP decoding.

For each $i = 1, \ldots, k$:

$$\hat{u}_i^{\text{symbol}}(\mathbf{y}) = \underset{u_i \in \mathcal{U}}{\operatorname{argmax}} \, P_{U_i|\mathbf{Y}}(u_i|\mathbf{y}) = \underset{u_i \in \mathcal{U}}{\operatorname{argmax}} \, P_{U_i,\mathbf{Y}}(u_i, \mathbf{y}).$$

Symbol-Wise MAP Decoding (Part 2)

Rewriting symbol-wise MAP decoding for symbol i we obtain

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Binary Linear Codes

Let $\mathbf{H}$ be a parity-check matrix, e.g.

$$\mathbf{H} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

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$$C = \left\{ (x_1, x_2, x_3, x_4, x_5) \in \mathbb{F}_2^5 \;\middle|\; \mathbf{H} \cdot \mathbf{x}^\top = \mathbf{0}^\top \pmod 2 \right\}.$$

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A vector $\mathbf{x} \in \mathbb{F}_2^5$ is a codeword if and only if

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Binary Linear Codes

This means that $\mathbf{x}$ is a codeword if and only if $\mathbf{x}$ fulfills the following two equations:

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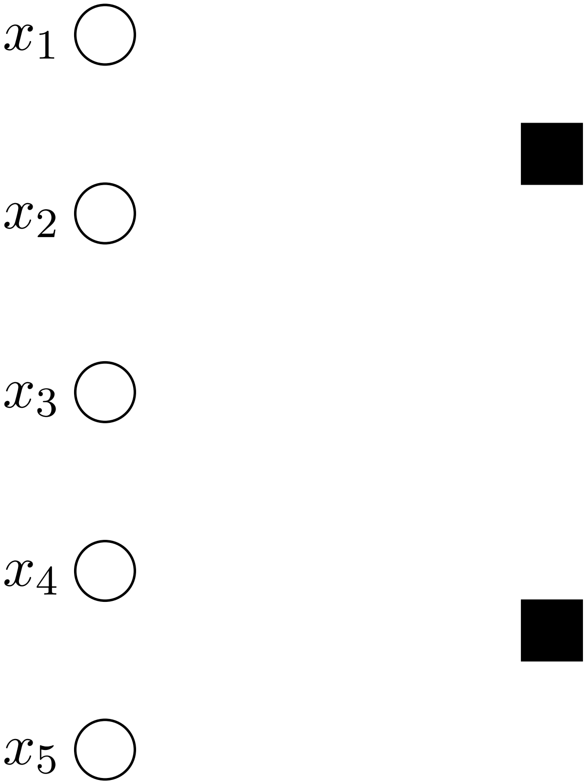
In summary,

$$\begin{aligned} C &= \left\{ (x_1, x_2, x_3, x_4, x_5) \in \mathbb{F}_2^5 \,\middle|\, \mathbf{H} \cdot \mathbf{x}^\top = \mathbf{0}^\top \pmod 2 \right\} \\ &= \left\{ (x_1, x_2, x_3, x_4, x_5) \in \mathbb{F}_2^5 \,\middle|\, \begin{aligned} x_1 + x_2 + x_3 &= 0 \pmod 2 \\ x_2 + x_4 + x_5 &= 0 \pmod 2 \end{aligned} \right\}. \end{aligned}$$

Graphical Representation of a Code

$$\mathbf{H} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$



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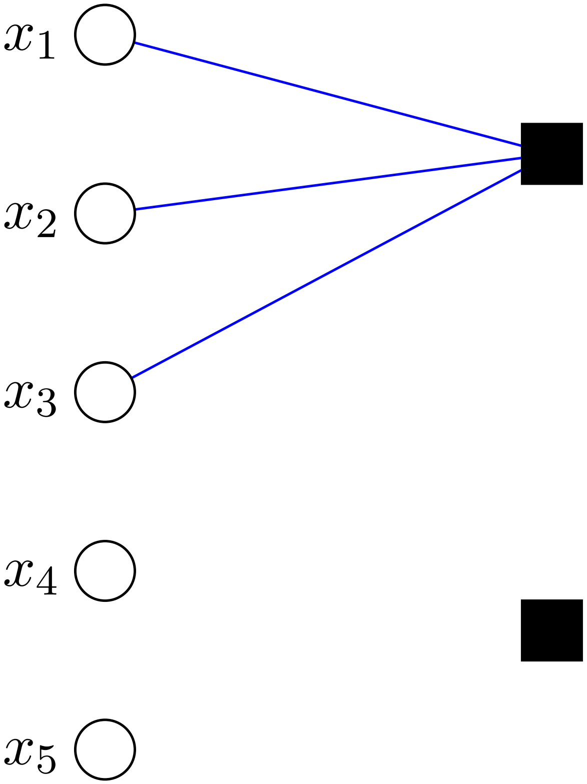

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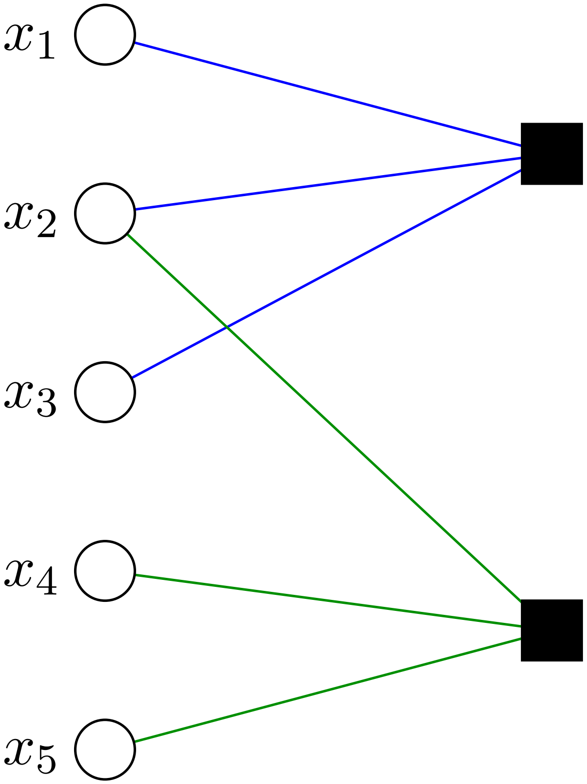

FG of a Data Communication System based on a parity-check code

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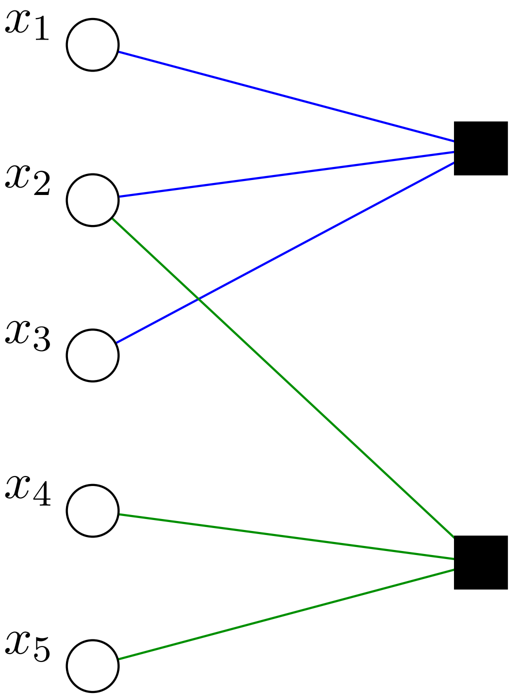

$$f_{\text{XOR}}(x_1, x_2, x_3) \;=\; [x_1 + x_2 + x_3 = 0 \pmod 2]$$

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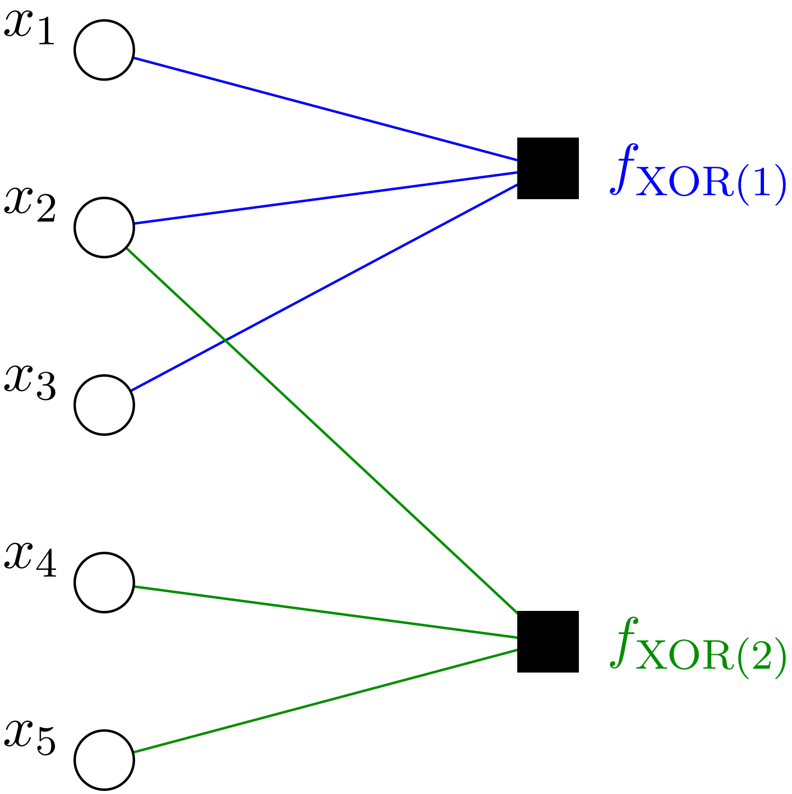

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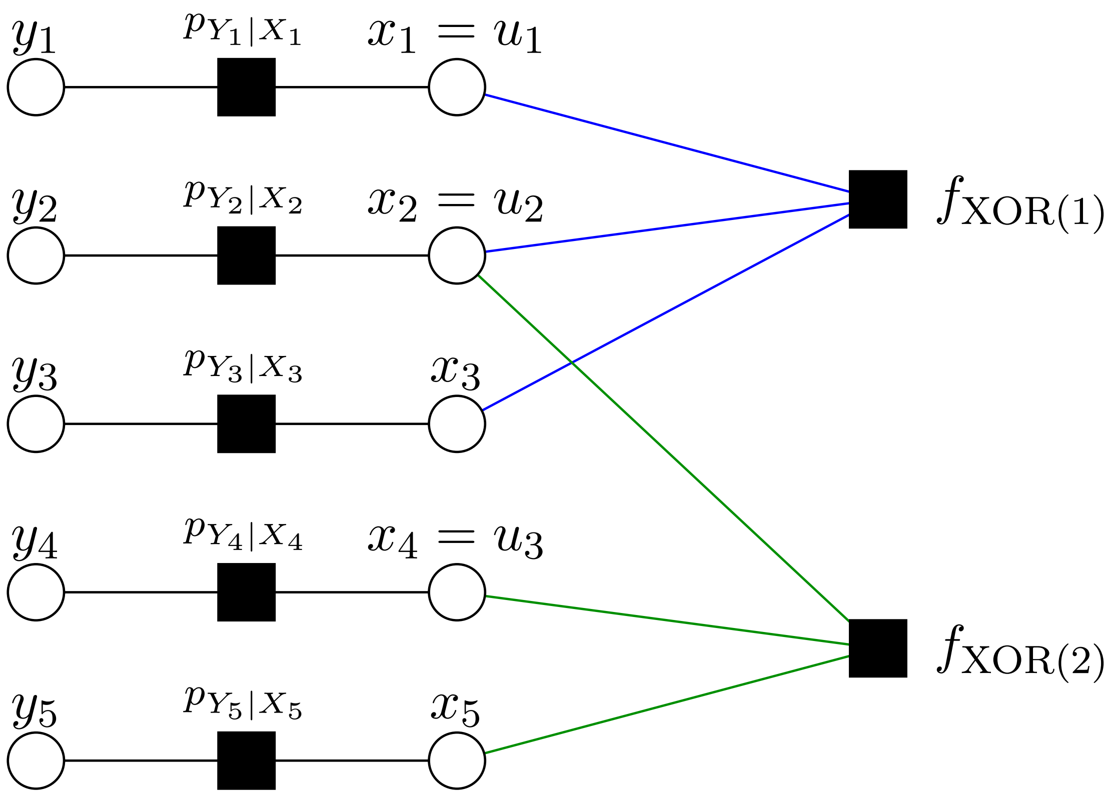
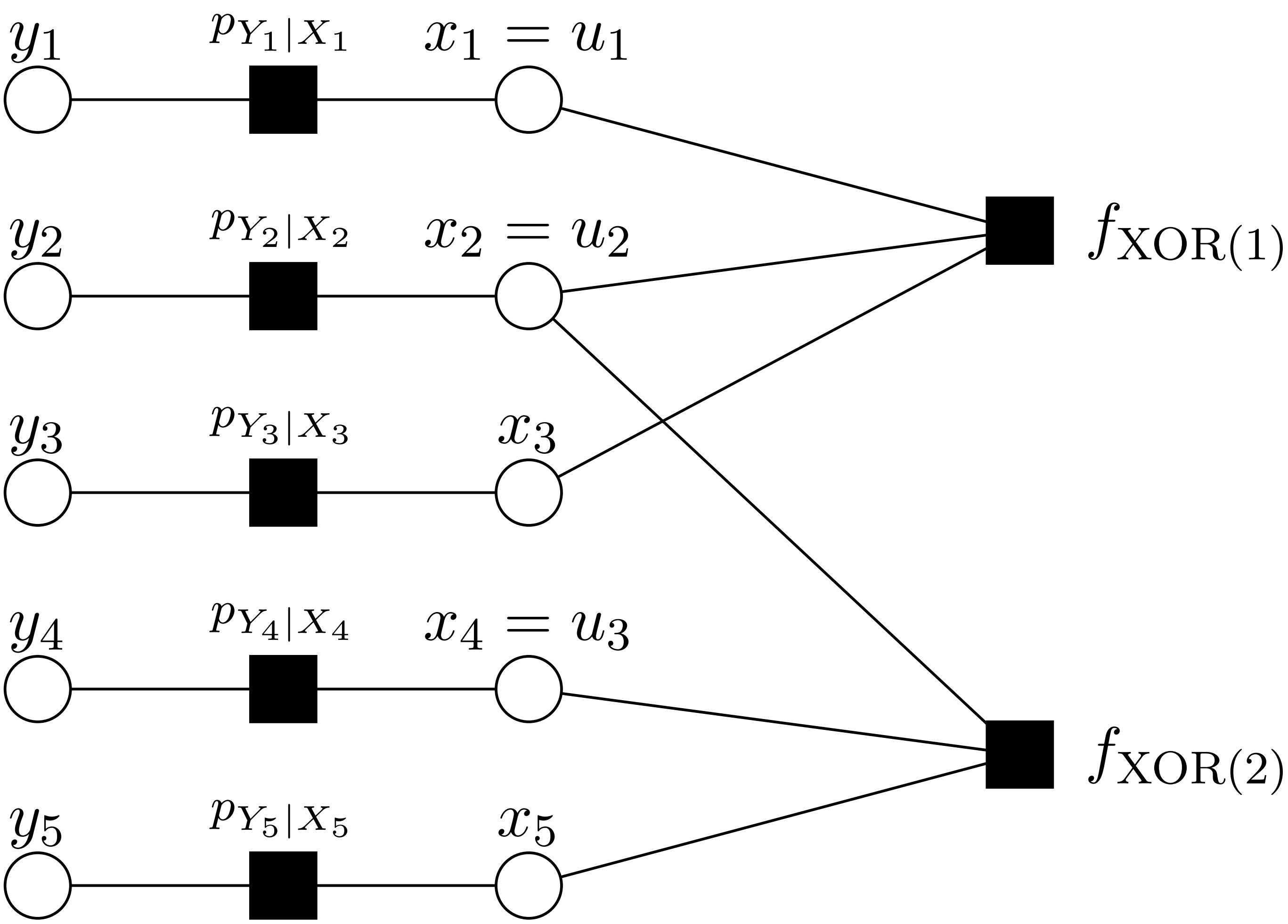


Symbol-Wise MAP Decoding

Remember, symbol-wise MAP decoding for symbol i can be written as

$$\hat{u}_i^{\text{symbol}}(\mathbf{y}) = \underbrace{\underbrace{\operatorname*{argmax}_{u_i\in\mathcal{U}}}_{\text{Decision taking}} \underbrace{\sum_{\substack{\mathbf{u}\in\mathcal{U}^k,\ \mathbf{x}\in\mathcal{X}^n\\ u_i\ \text{fixed}}} \underbrace{P_{\mathbf{UXY}}(\mathbf{u},\mathbf{x},\mathbf{y}),}_{\text{Joint pmf/pdf}}}_{\text{Marginal function}}}_{\text{Decision about symbol } u_i \text{ based on symbol-wise decoding}}$$

SPA Decoding (Factor graph without cycles)



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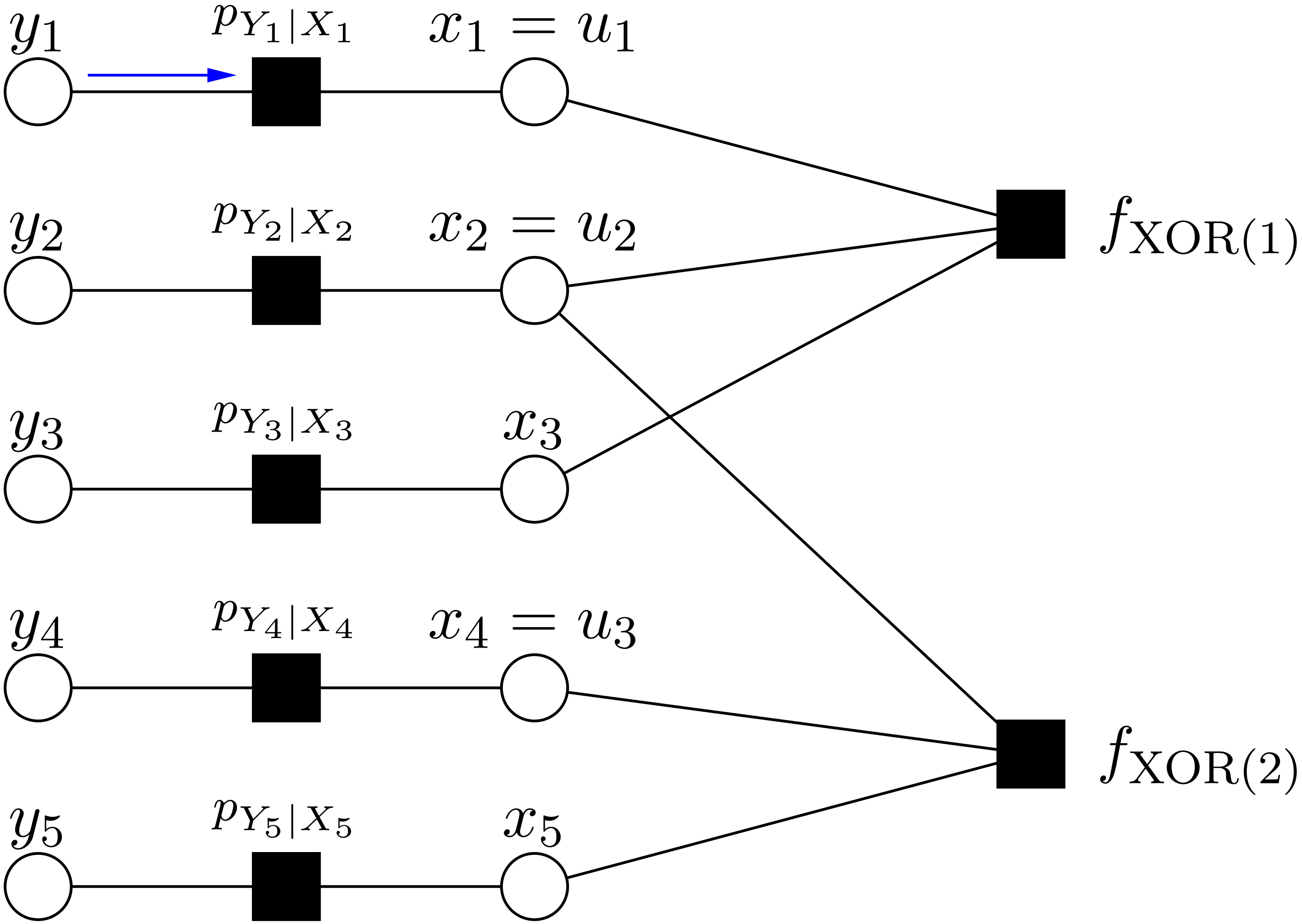

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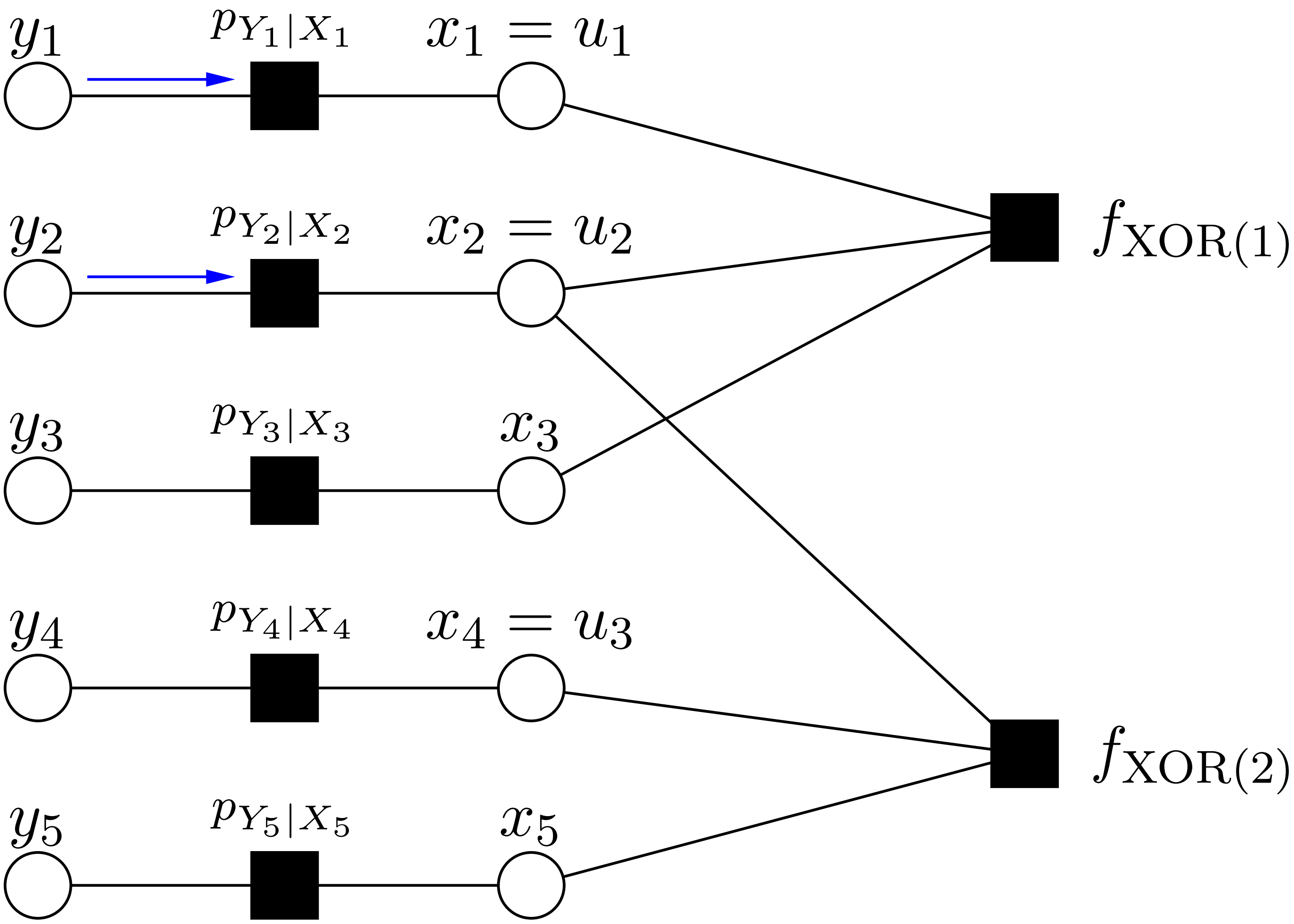

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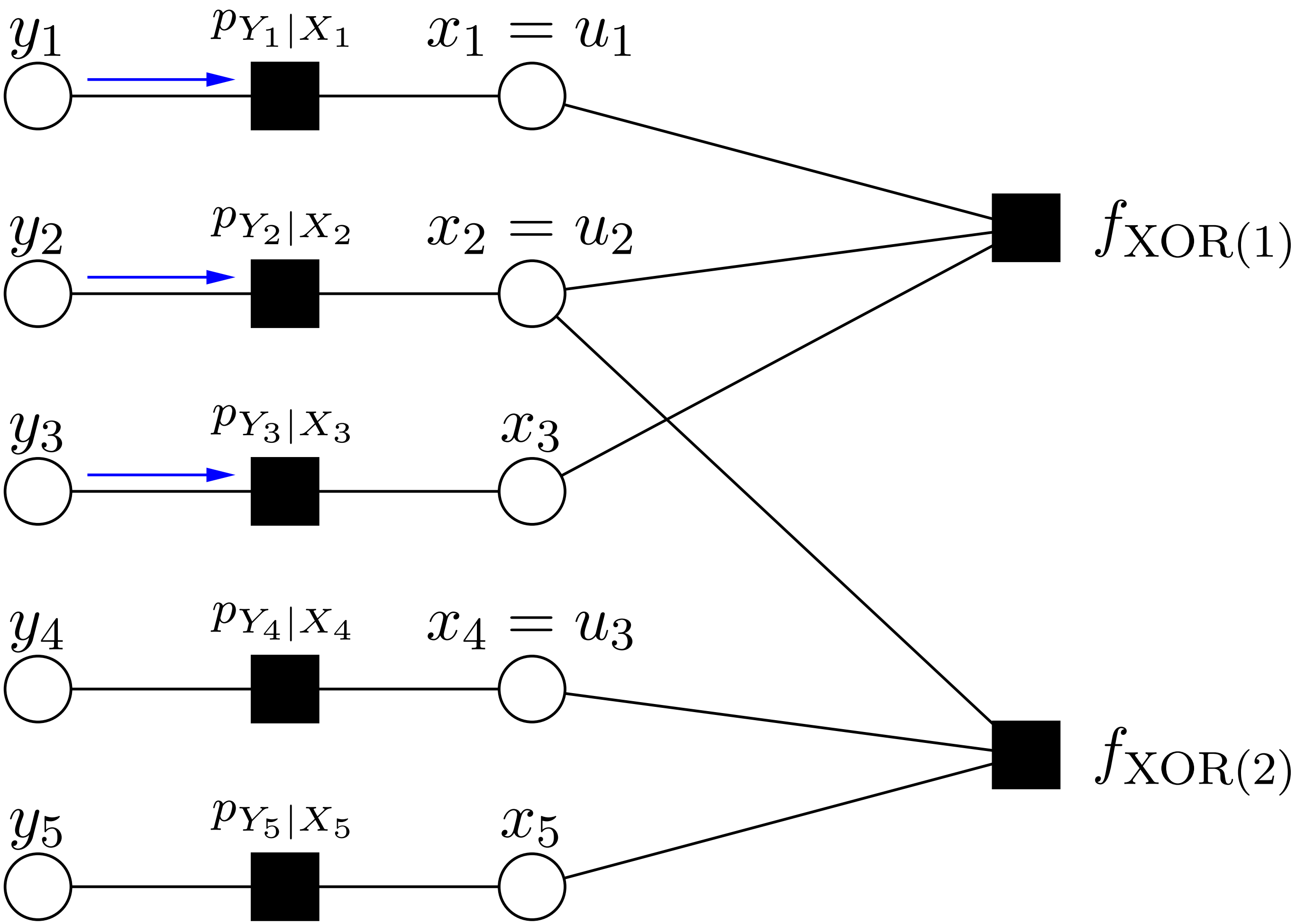

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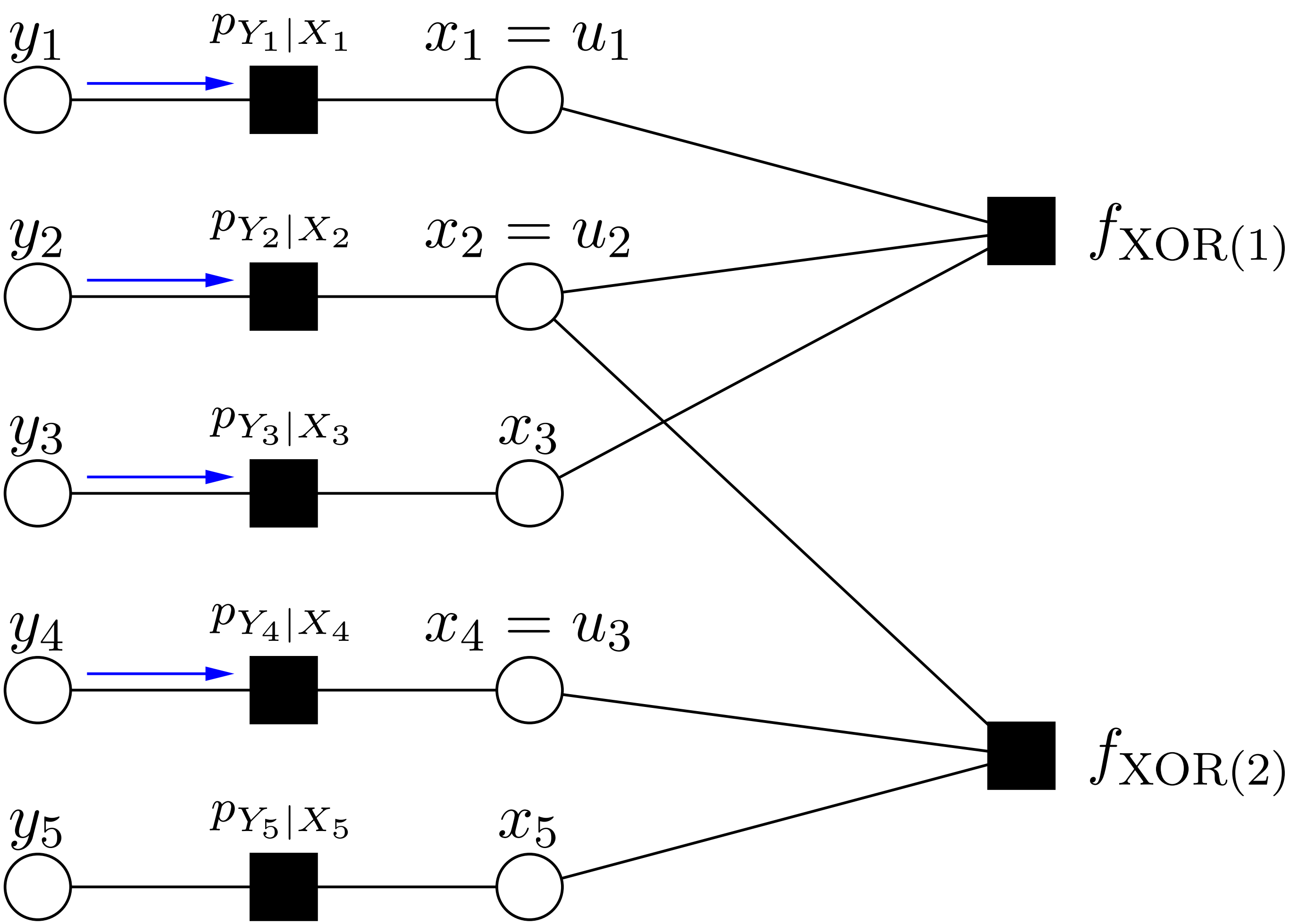

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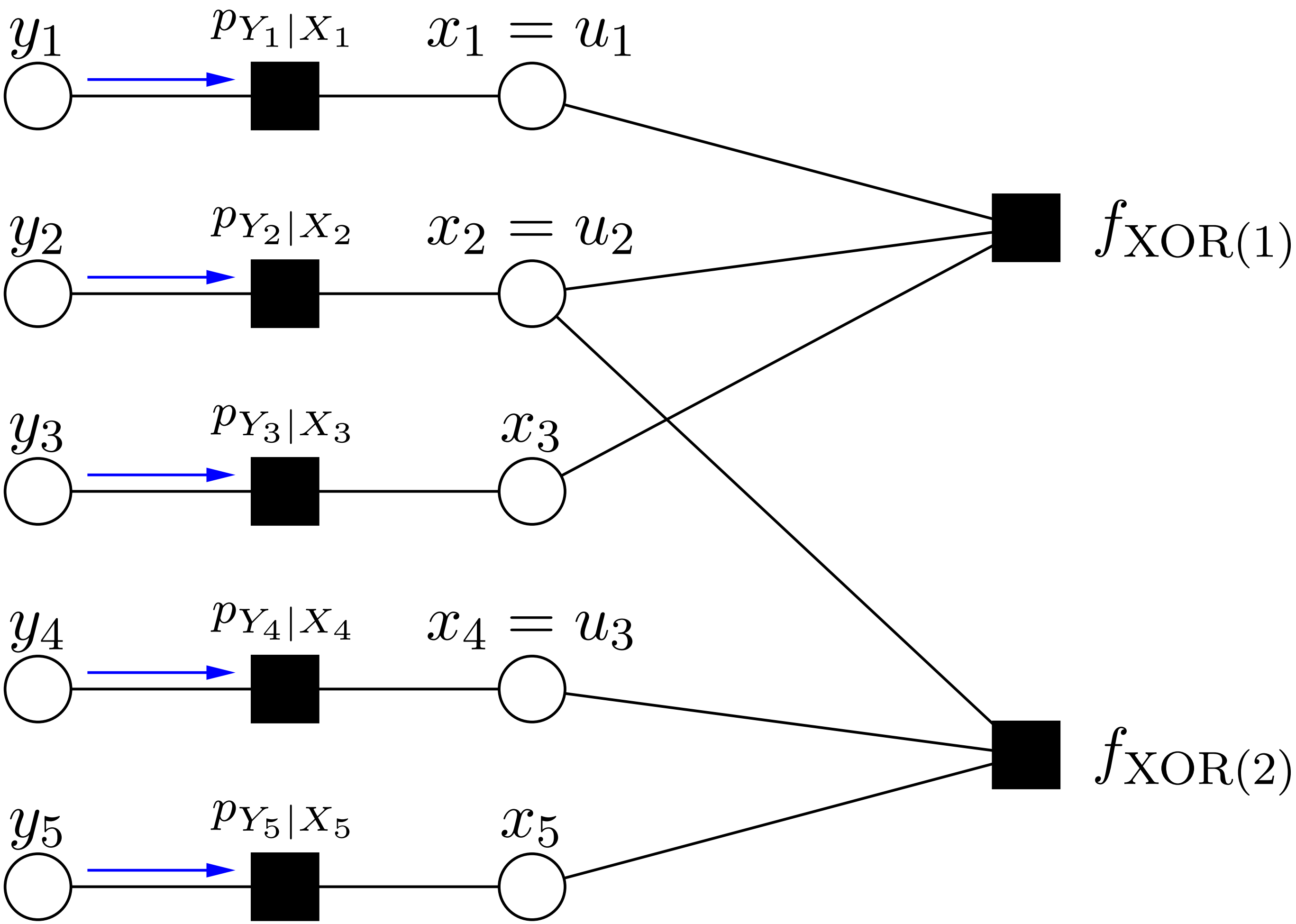

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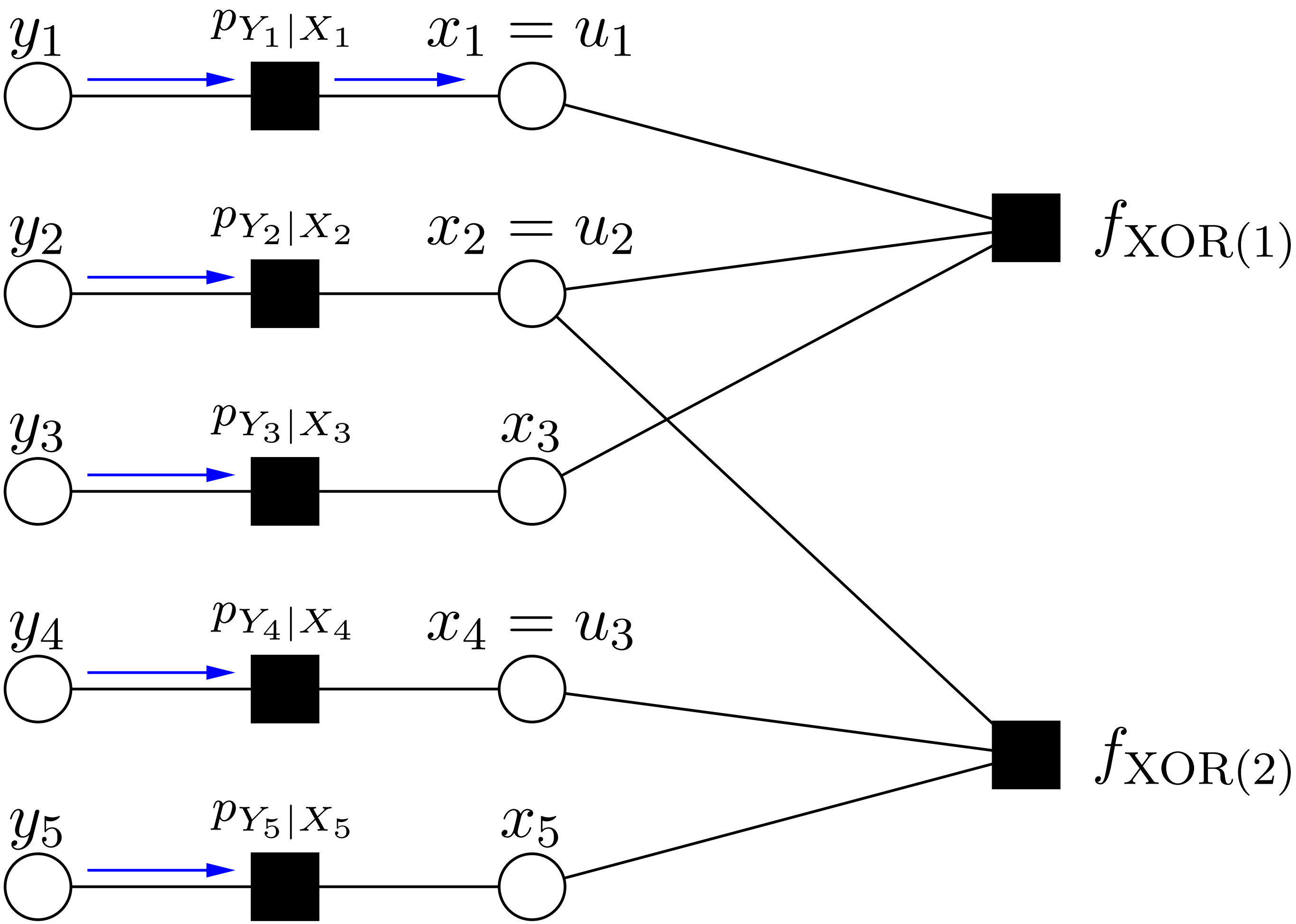

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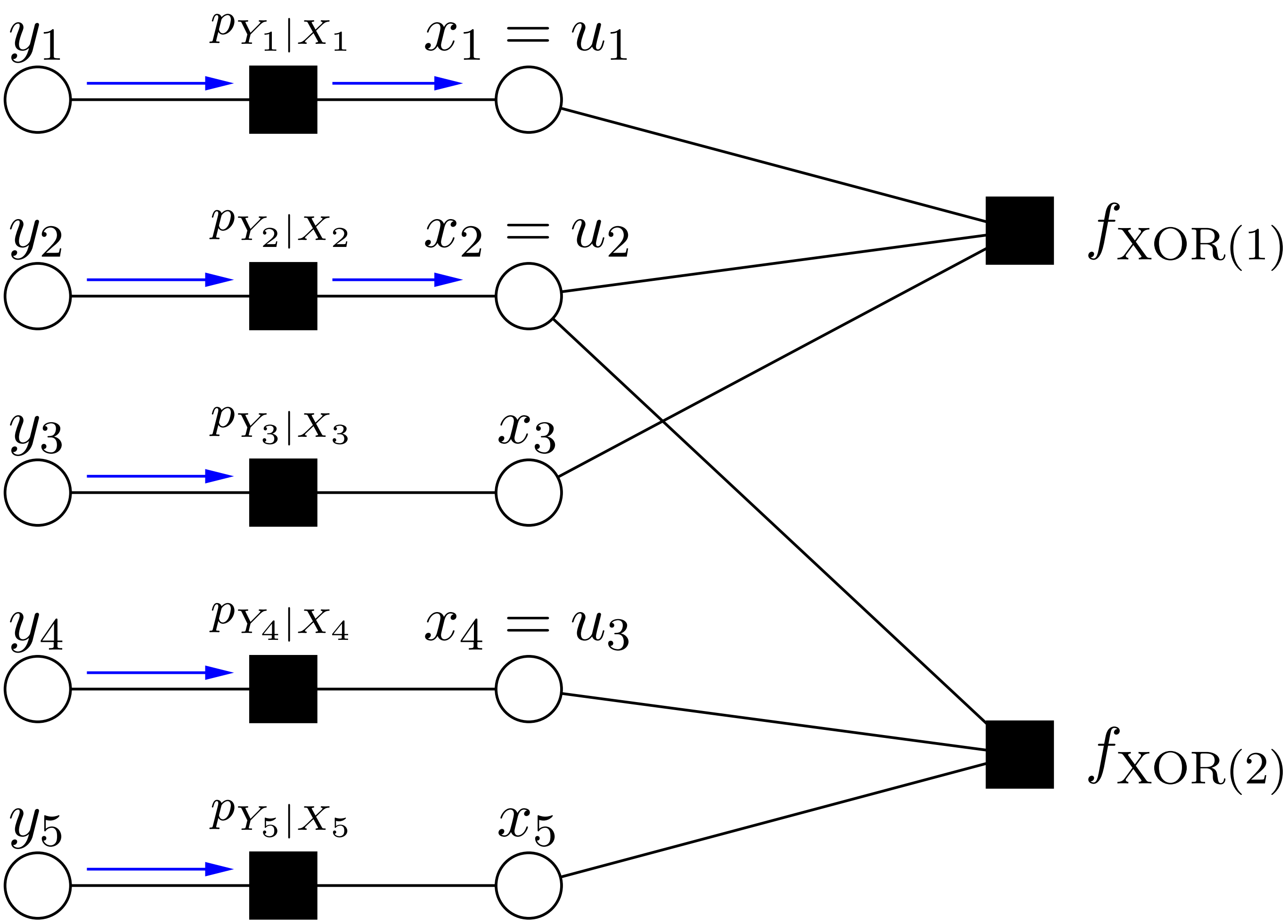

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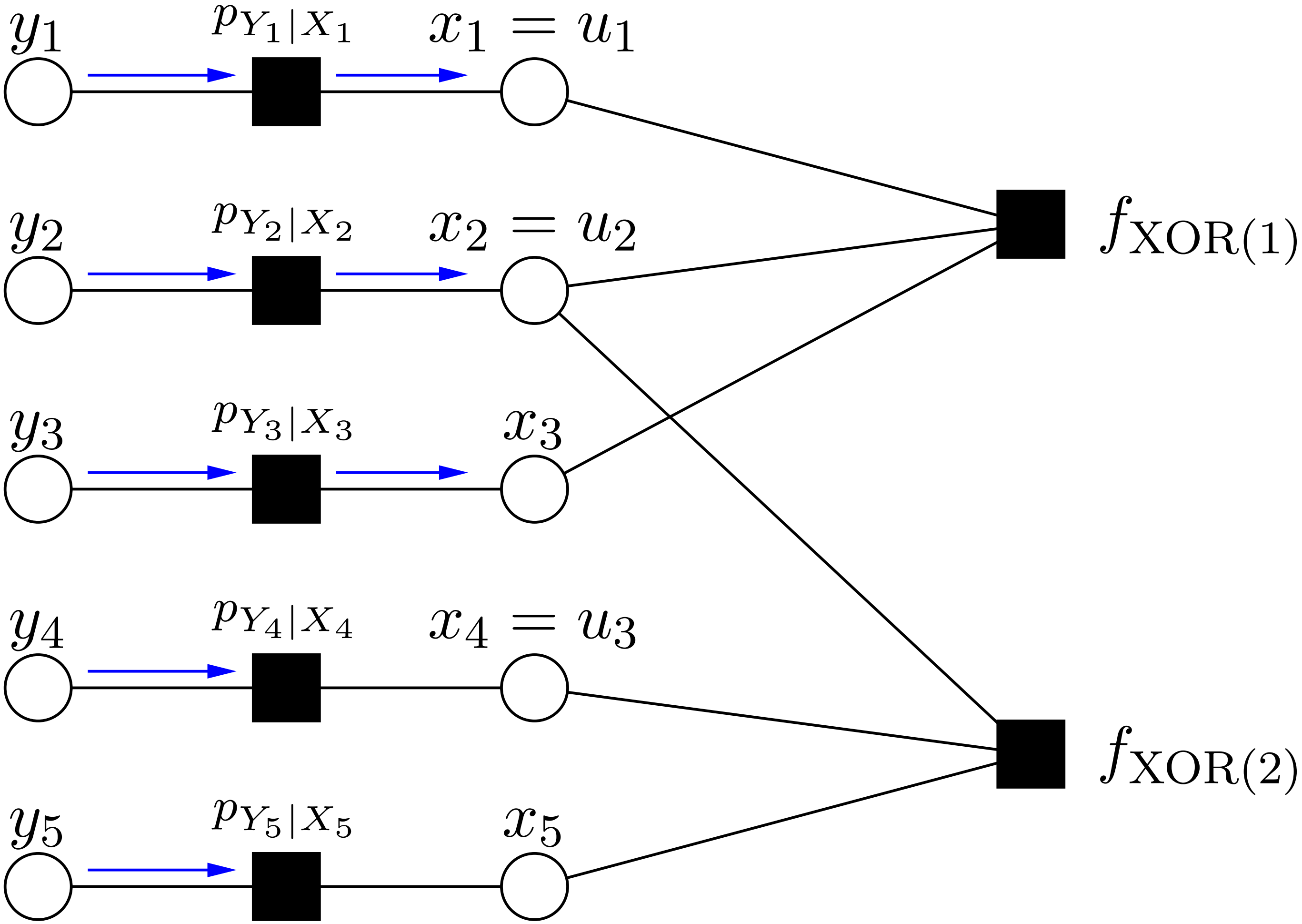

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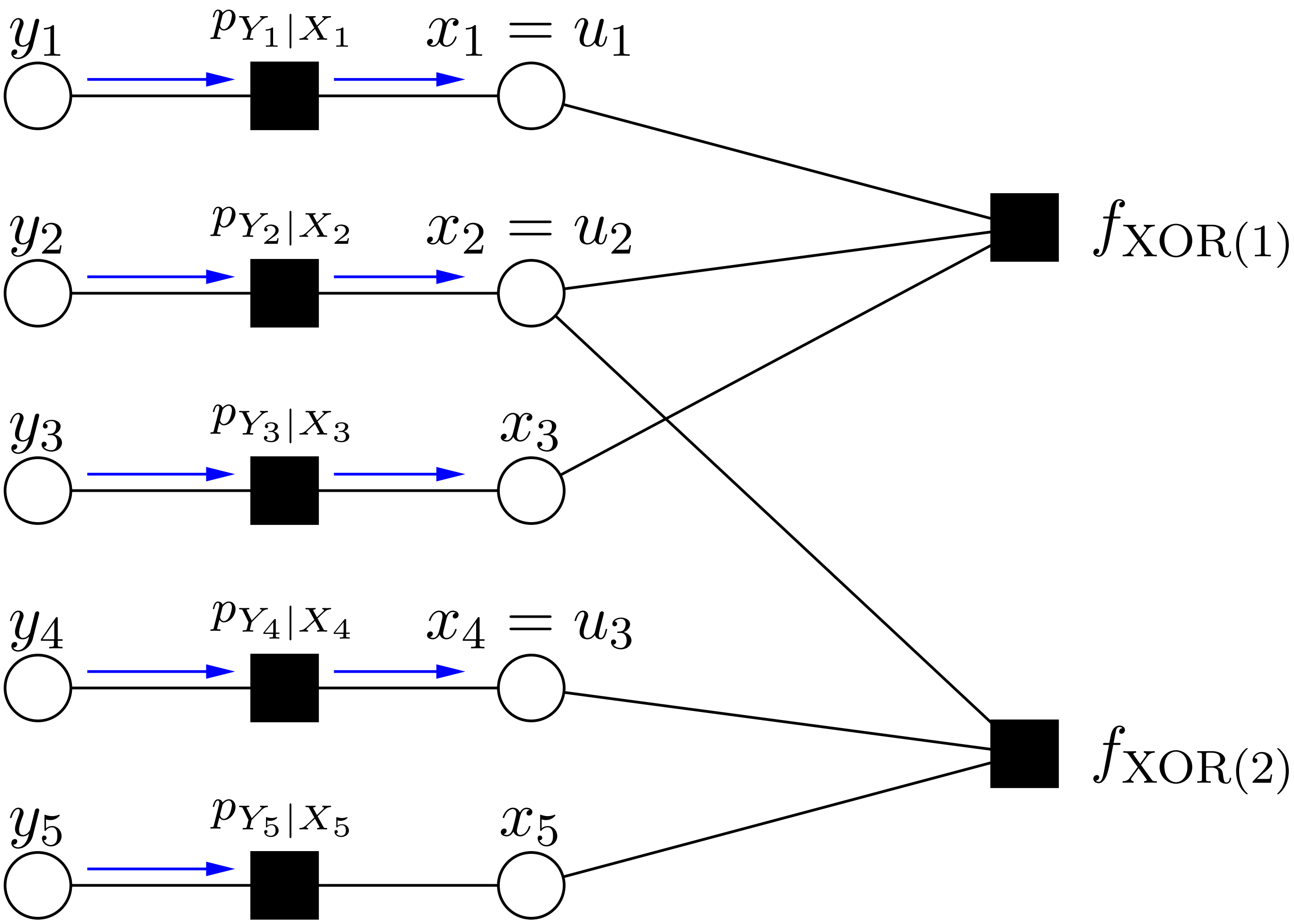

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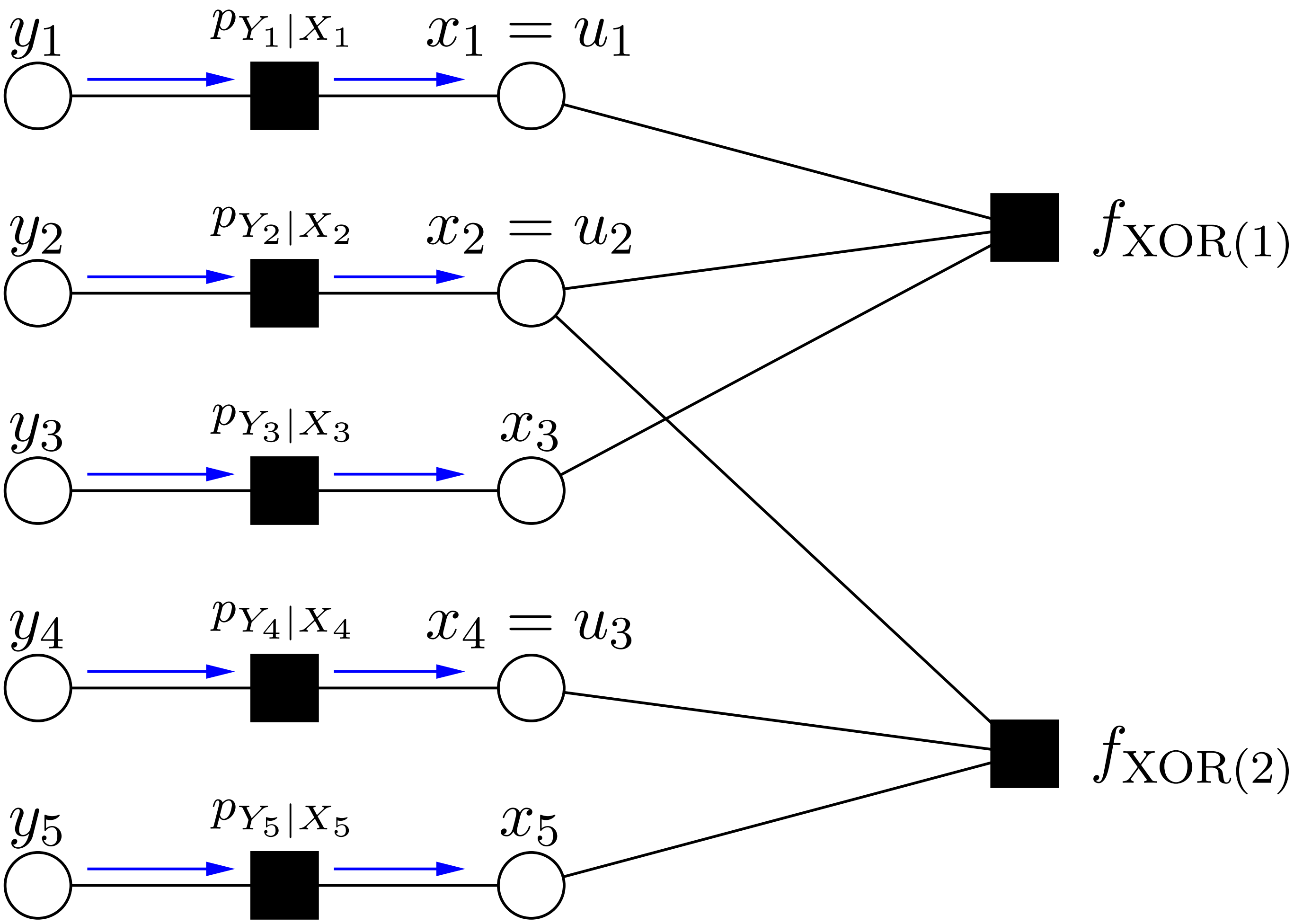

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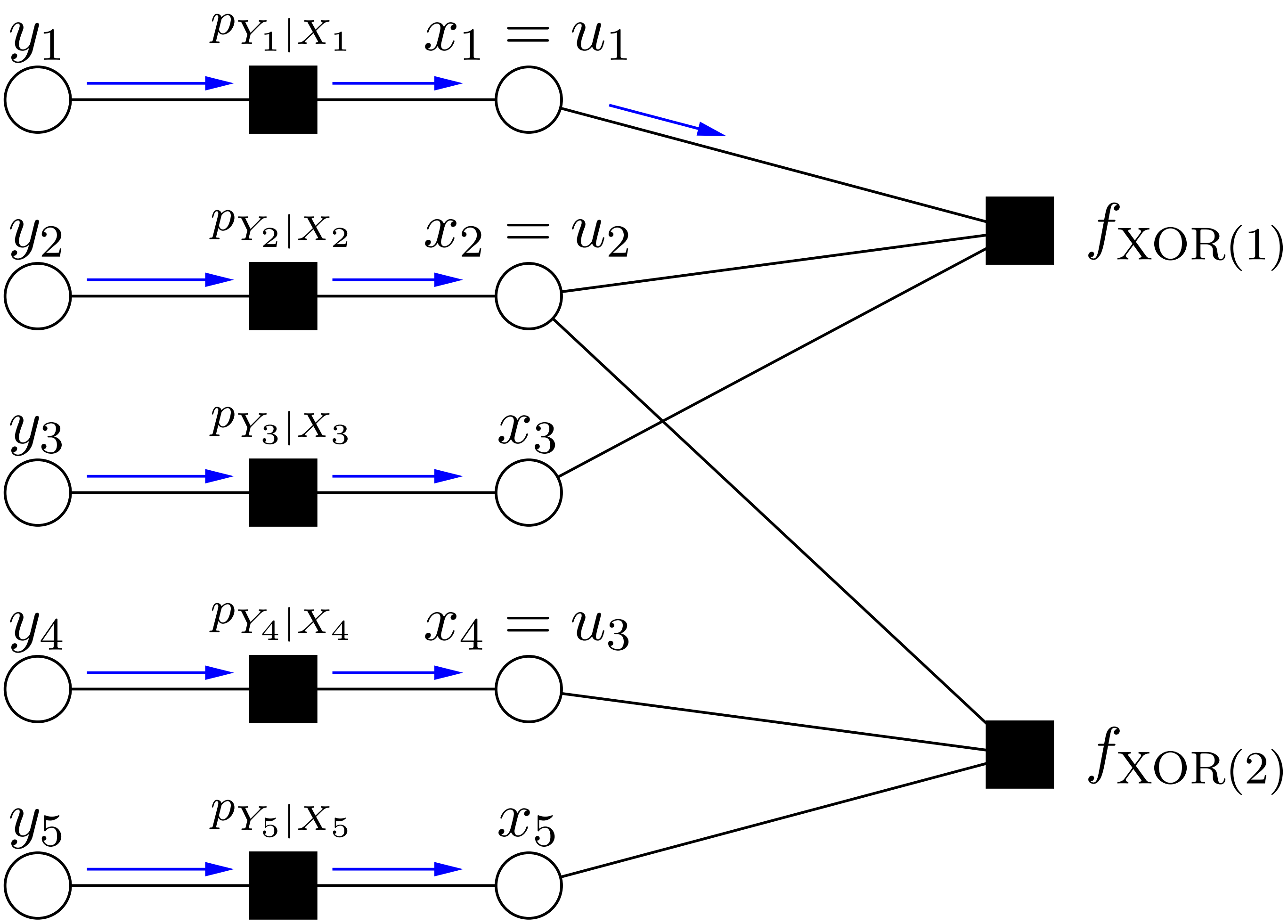

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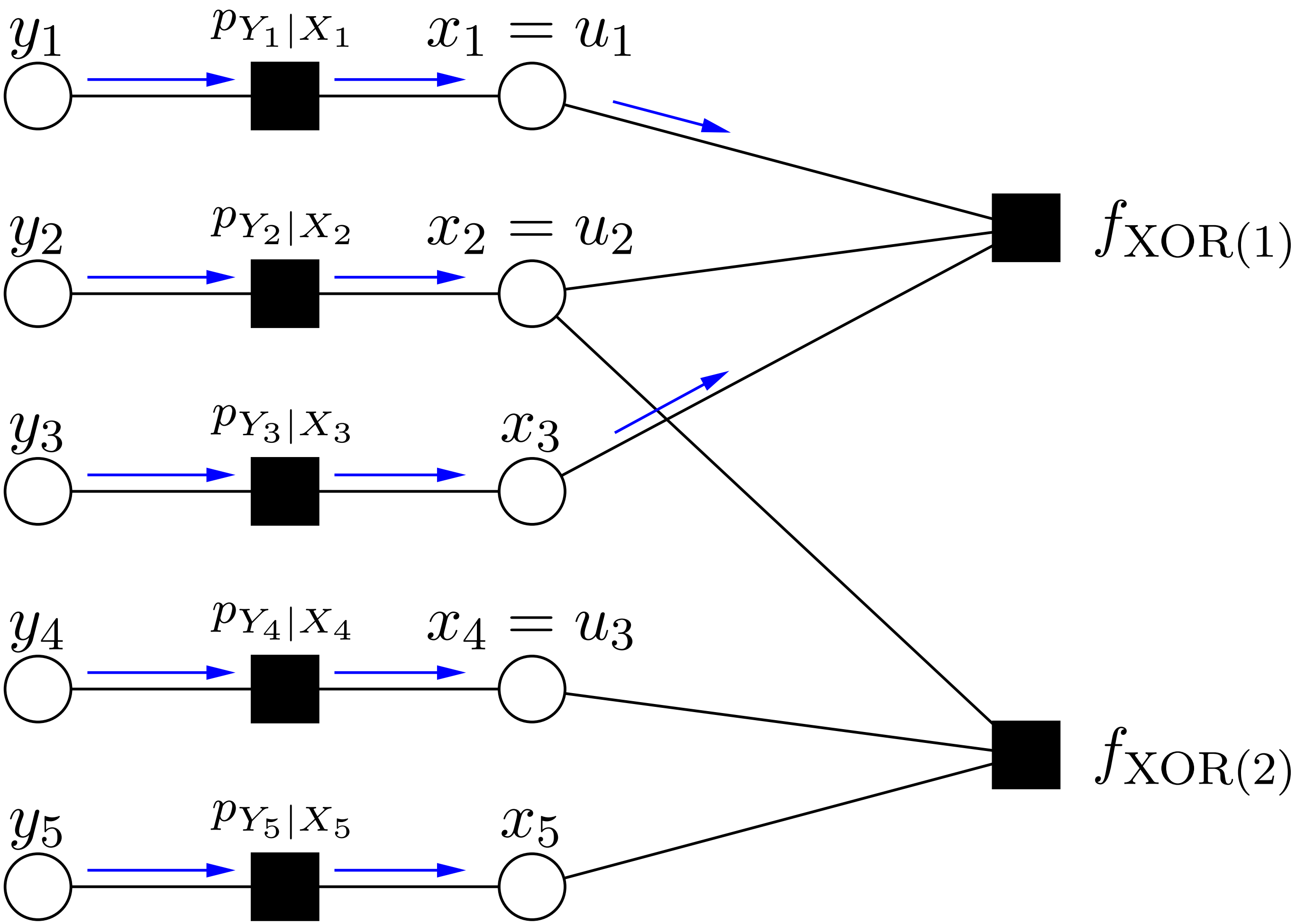

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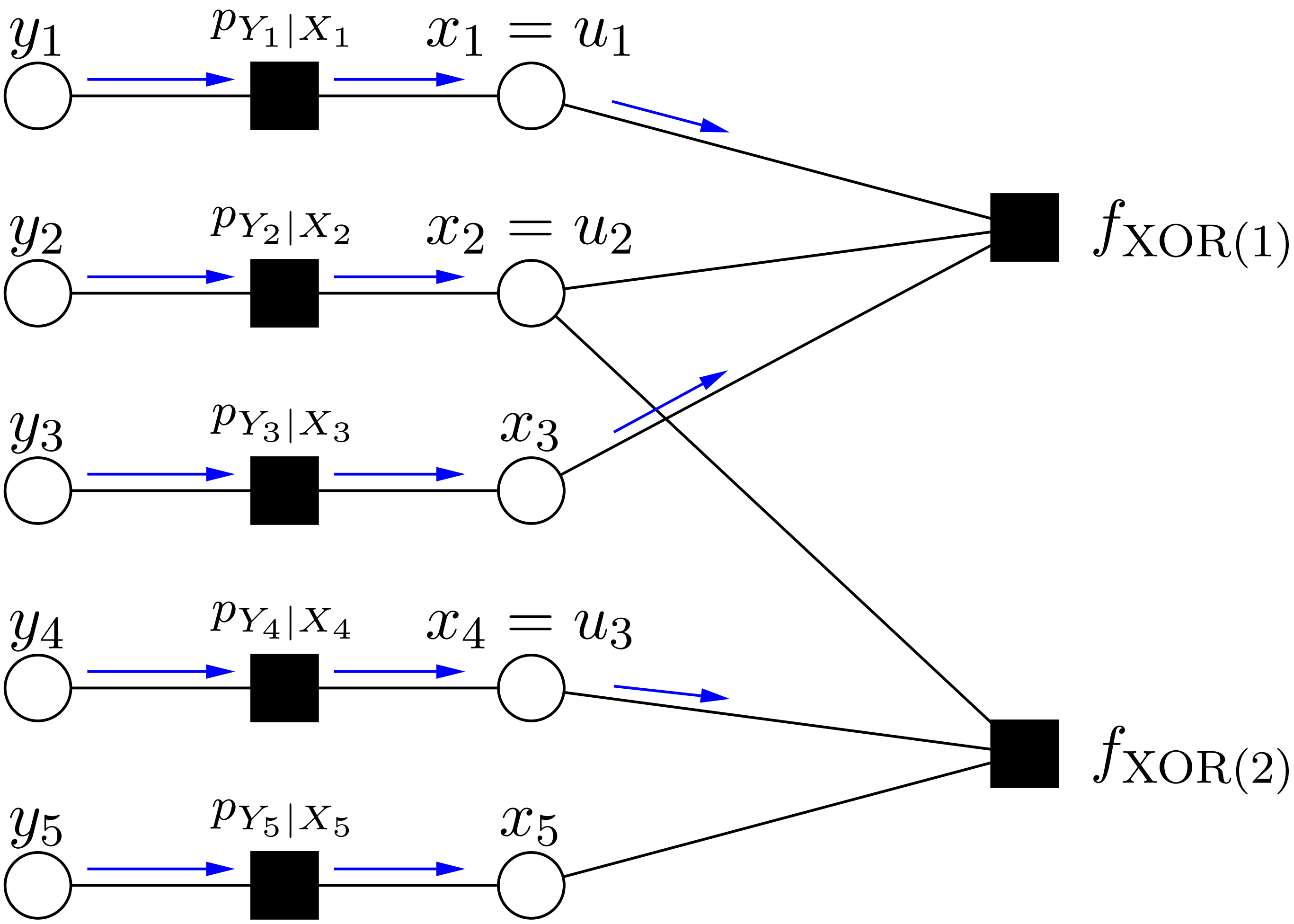

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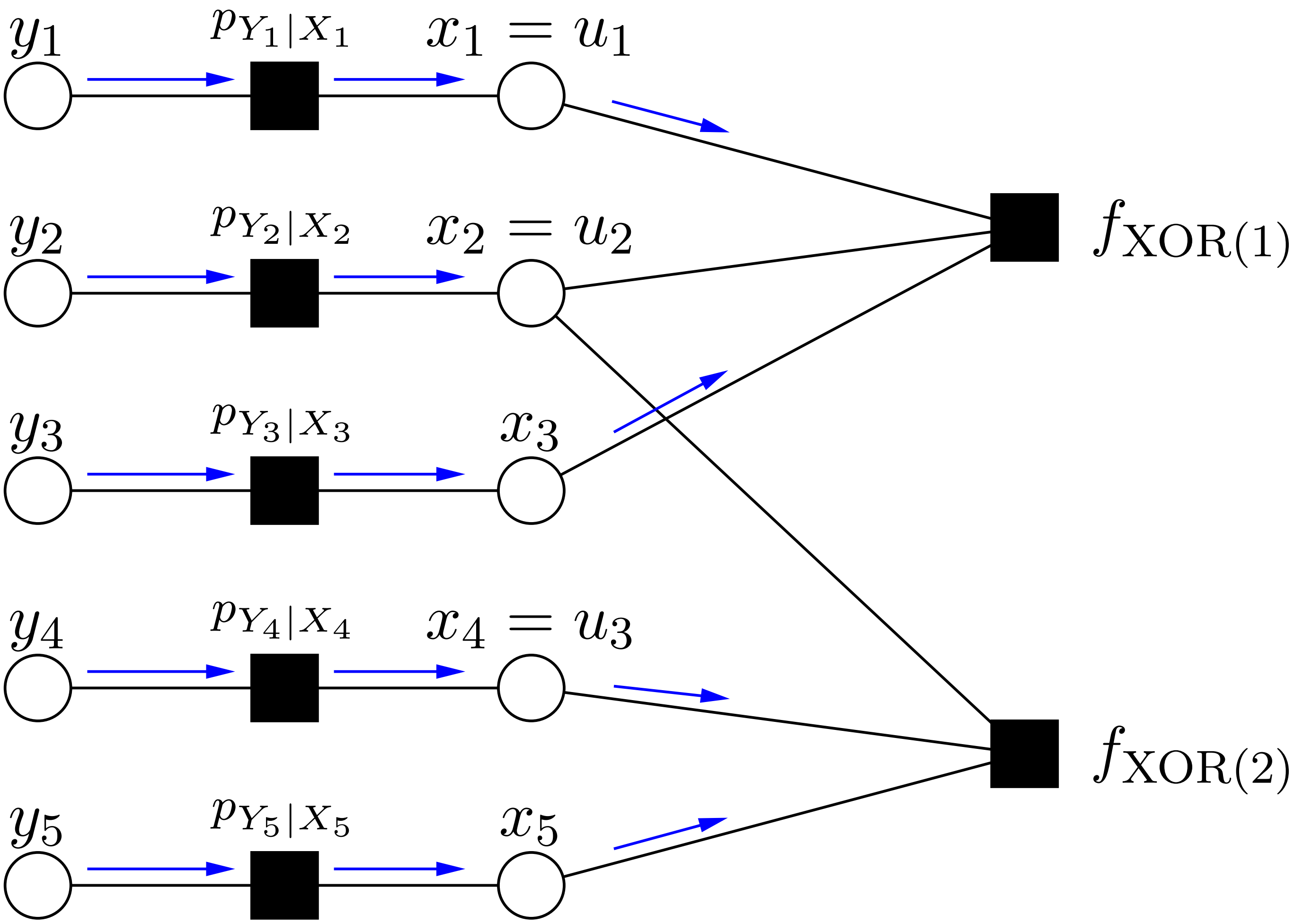

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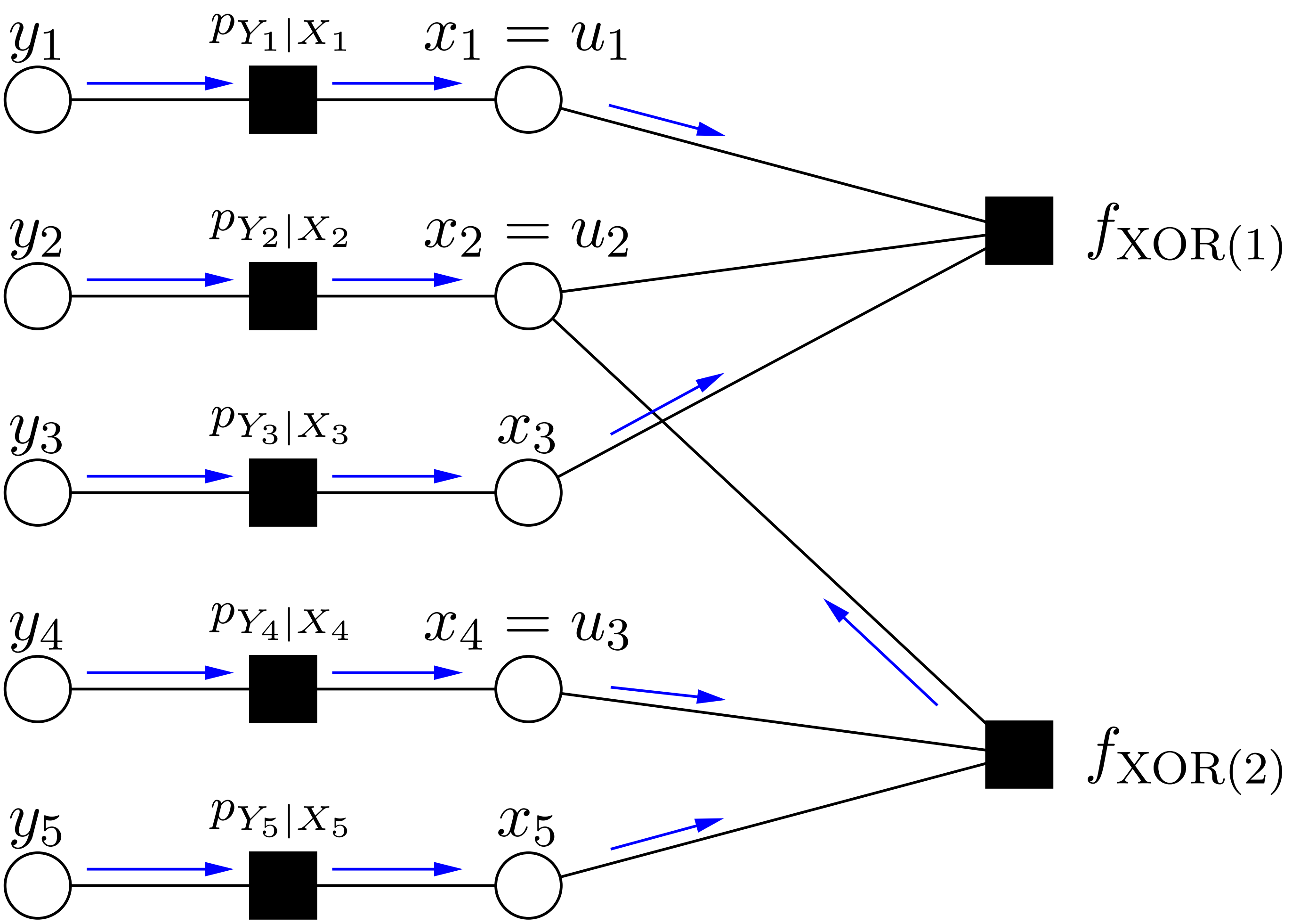

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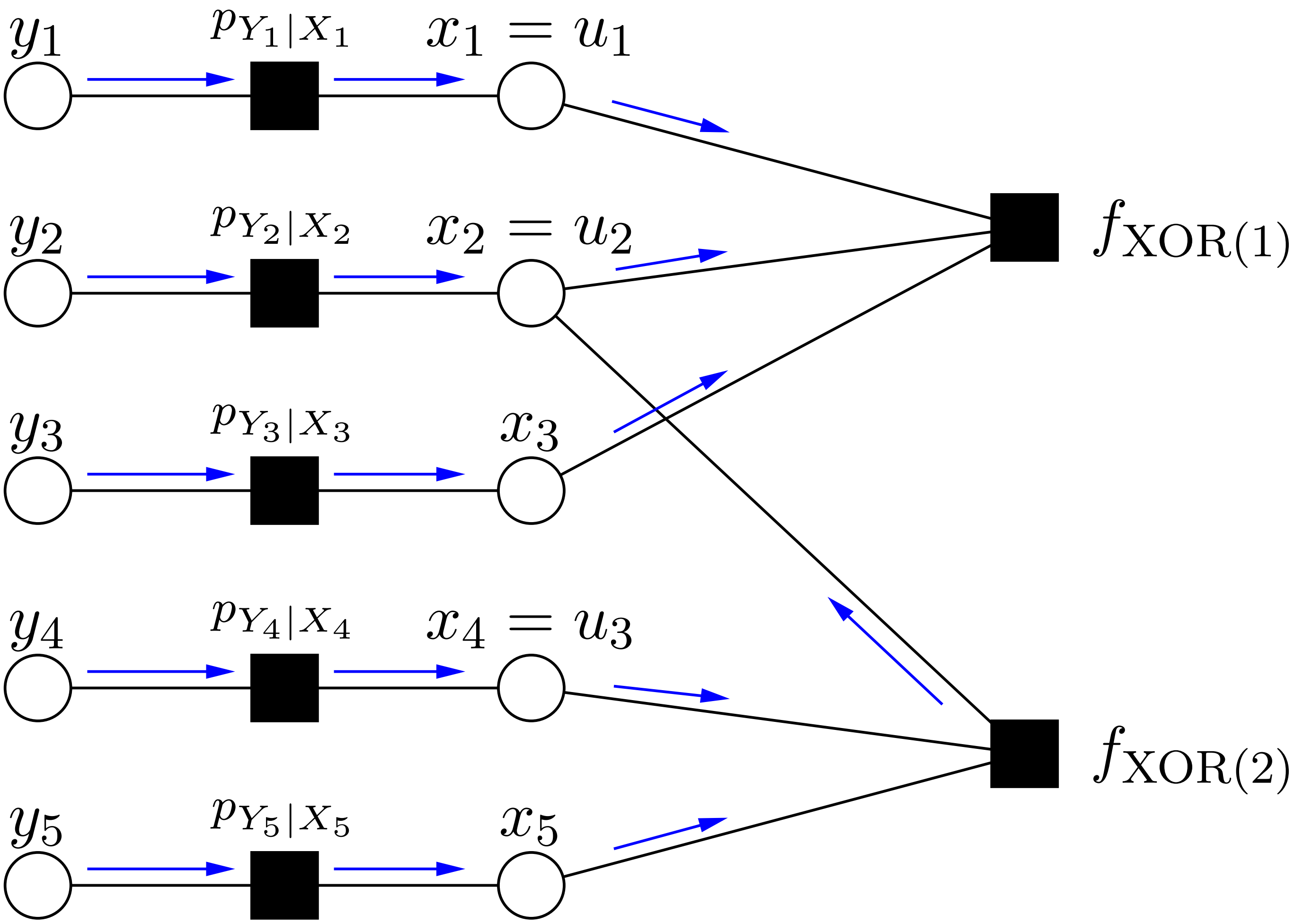

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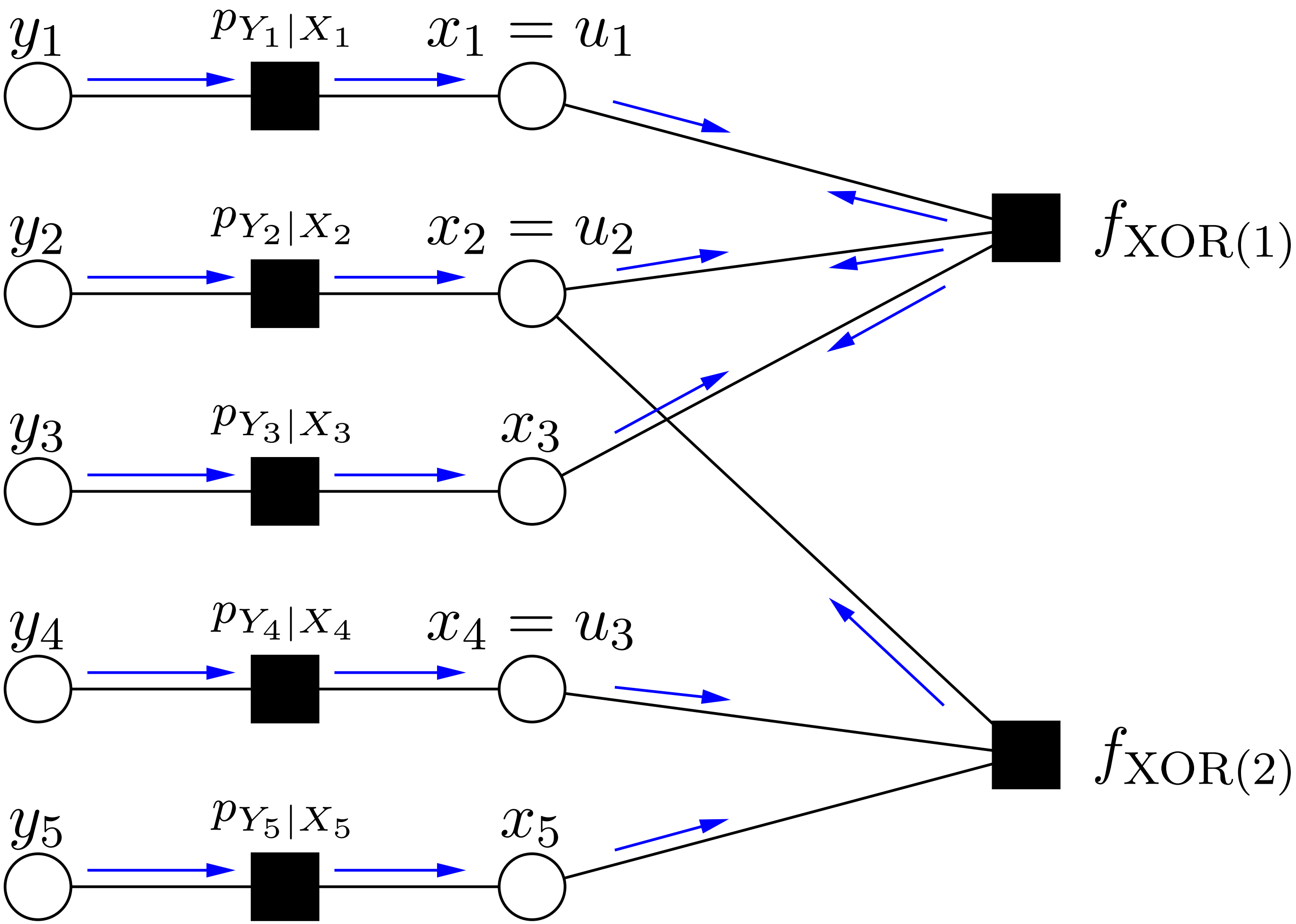

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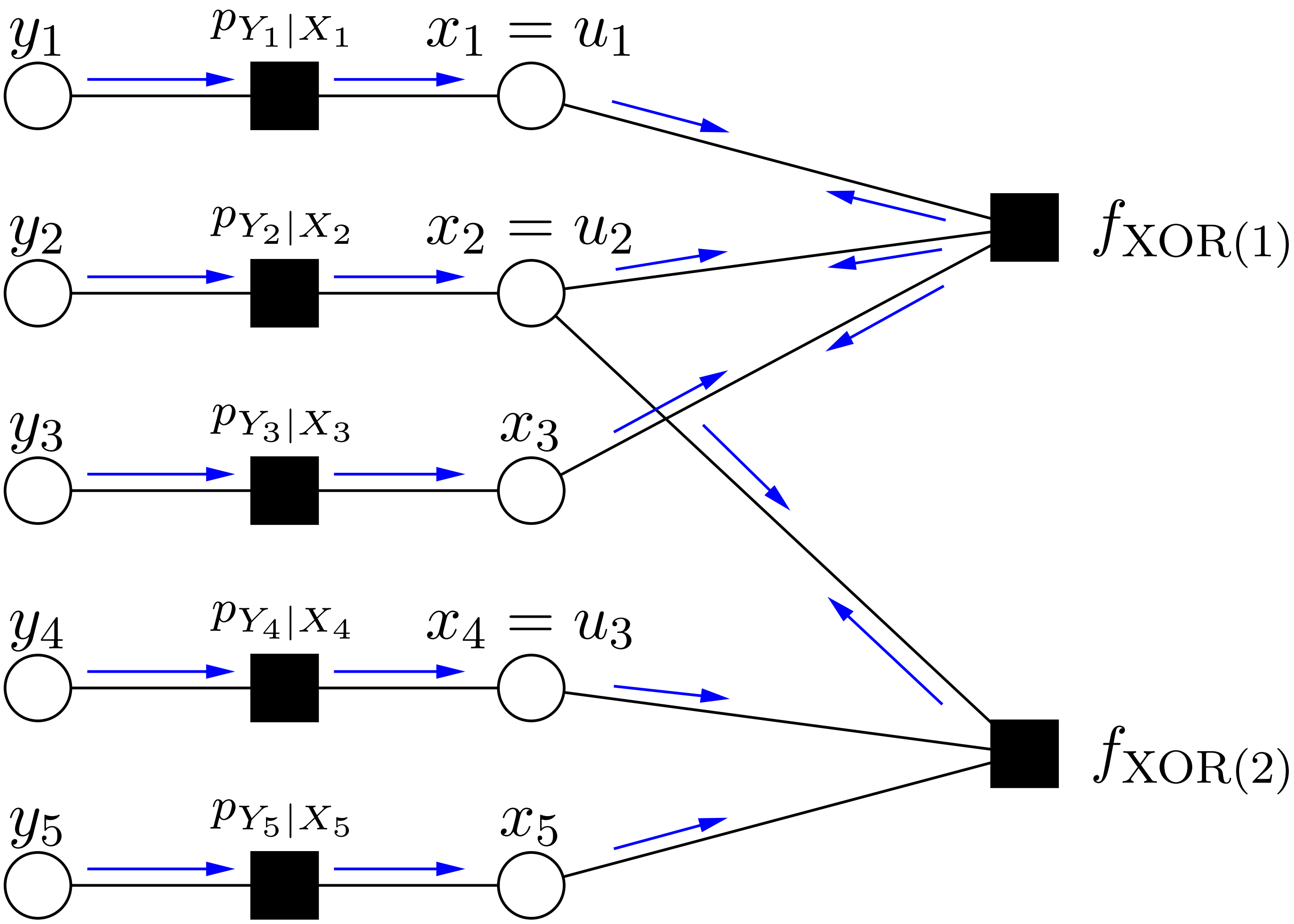
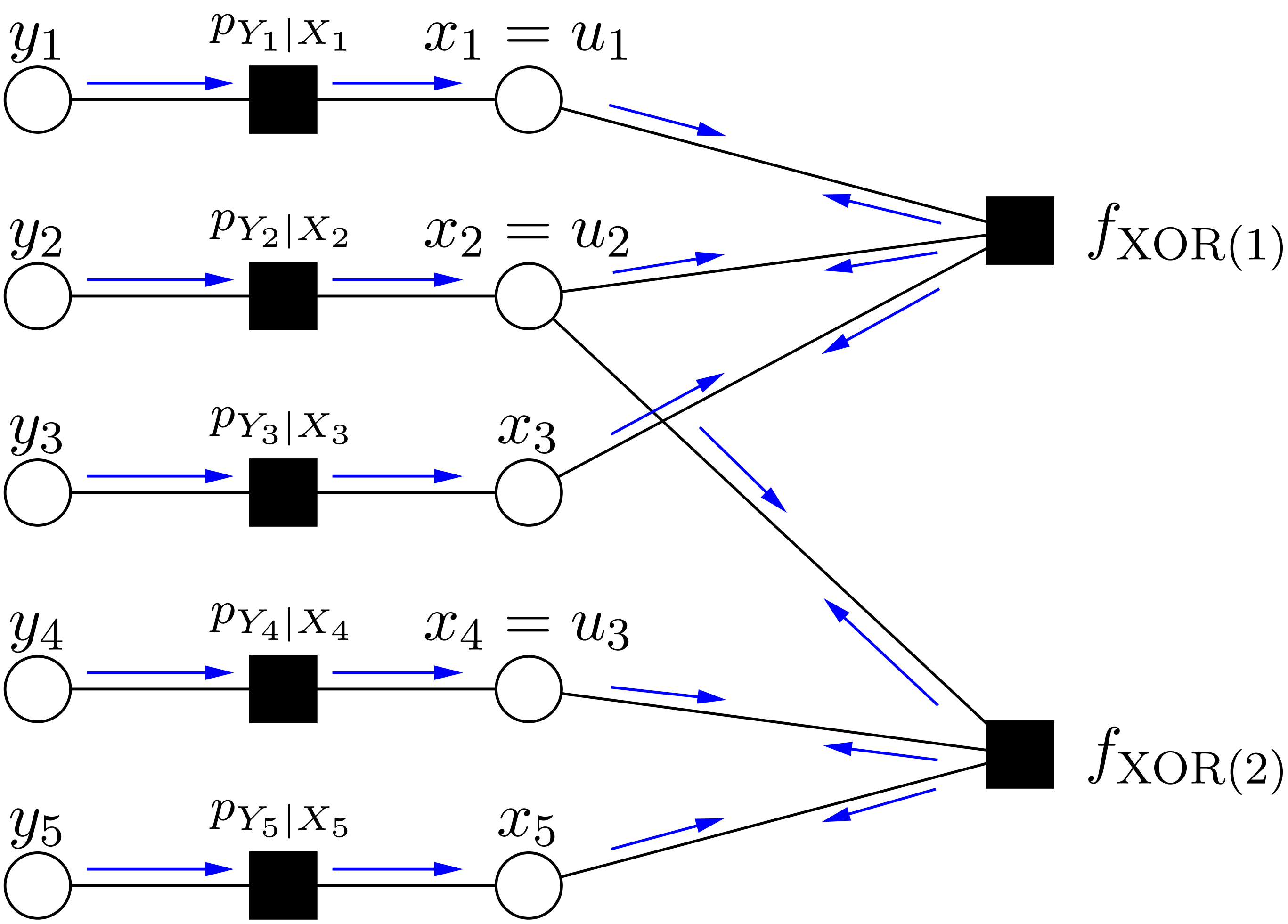
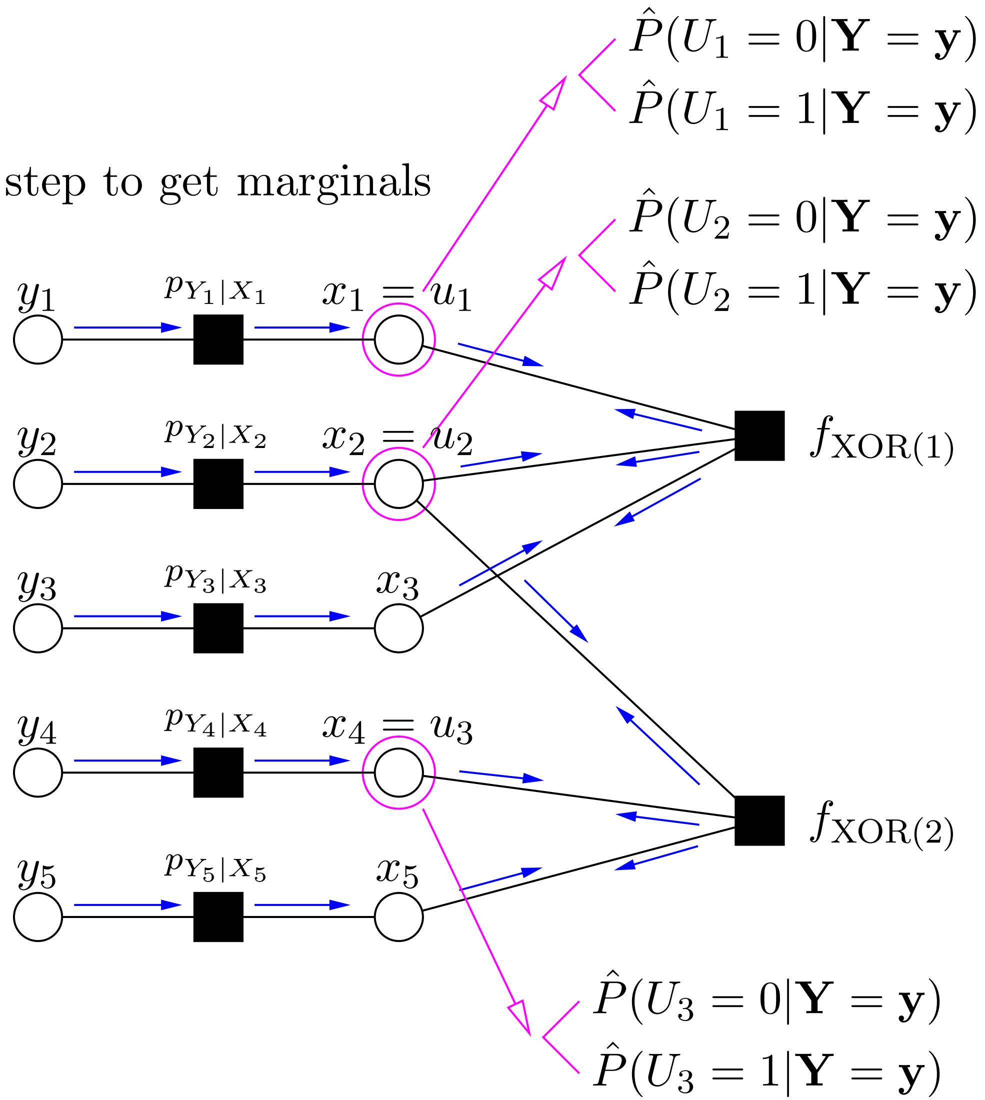


SPA Decoding (Factor graph without cycles)



SPA Decoding (Factor graph without cycles)


$\hat{P}(U_1 = 0|\mathbf{Y} = \mathbf{y})$
$\hat{P}(U_1 = 1|\mathbf{Y} = \mathbf{y})$
step to get marginals
$\hat{P}(U_2 = 0|\mathbf{Y} = \mathbf{y})$
$\hat{P}(U_2 = 1|\mathbf{Y} = \mathbf{y})$
y_1
$p_{Y_1|X_1}$
$x_1 = u_1$
y_2
$p_{Y_2|X_2}$
$x_2 = u_2$
$f_{\text{XOR}(1)}$
y_3
$p_{Y_3|X_3}$
x_3
y_4
$p_{Y_4|X_4}$
$x_4 = u_3$
$f_{\text{XOR}(2)}$
y_5
$p_{Y_5|X_5}$
x_5
$\hat{P}(U_3 = 0|\mathbf{Y} = \mathbf{y})$
$\hat{P}(U_3 = 1|\mathbf{Y} = \mathbf{y})$

Symbol-Wise MAP Decoding

Remember, symbol-wise MAP decoding for symbol i can be written as

$$\hat{u}_i^{\text{symbol}}(\mathbf{y}) = \underbrace{\underbrace{\operatorname*{argmax}_{u_i \in \mathcal{U}}}_{\text{Decision taking}} \underbrace{\sum_{\substack{\mathbf{u}\in\mathcal{U}^k,\ \mathbf{x}\in\mathcal{X}^n \\ u_i \text{ fixed}}} \underbrace{P_{\mathbf{UXY}}(\mathbf{u},\mathbf{x},\mathbf{y})}_{\text{Joint pmf/pdf}}}_{\text{Marginal function}},}_{\text{Decision about symbol } u_i \text{ based on symbol-wise decoding}}$$

Sum-Product Algorithm Decoding

Sum-product algorithm (SPA) decoding:

$$\hat{u}_i^{\text{symbol}}(\mathbf{y}) \approx \underbrace{\underbrace{\operatorname*{argmax}_{u_i\in\mathcal{U}}}_{\text{Decision taking}} \underbrace{\sum_{\substack{\mathbf{u}\in\mathcal{U}^k,\ \mathbf{x}\in\mathcal{X}^n \\ u_i\ \text{fixed}}} \underbrace{P_{\mathbf{UXY}}(\mathbf{u},\mathbf{x},\mathbf{y}),}_{\text{Joint pmf/pdf}}}_{\text{Marginal function is approximated by Sum-Product Algorithm}}}_{\text{Decision about symbol } u_i \text{ based on symbol-wise decoding}}$$

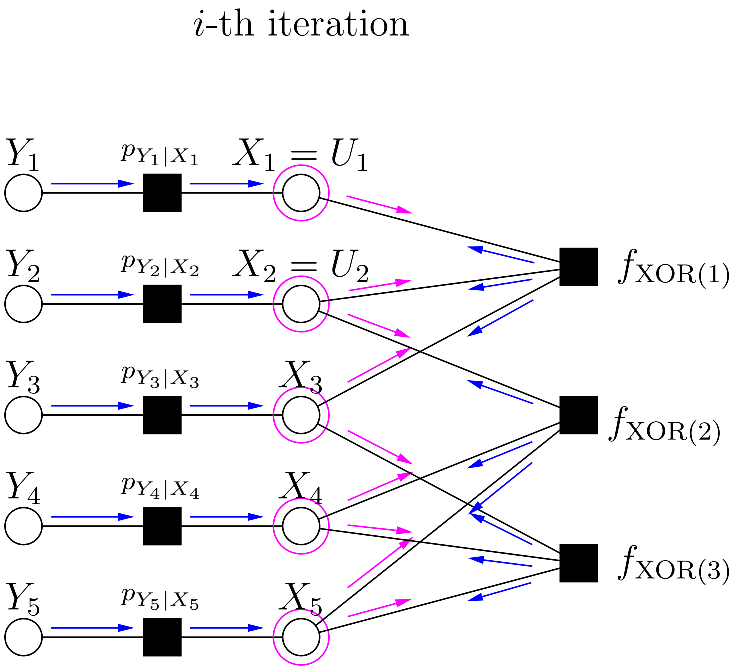
Sum-Product Algorithm Decoding

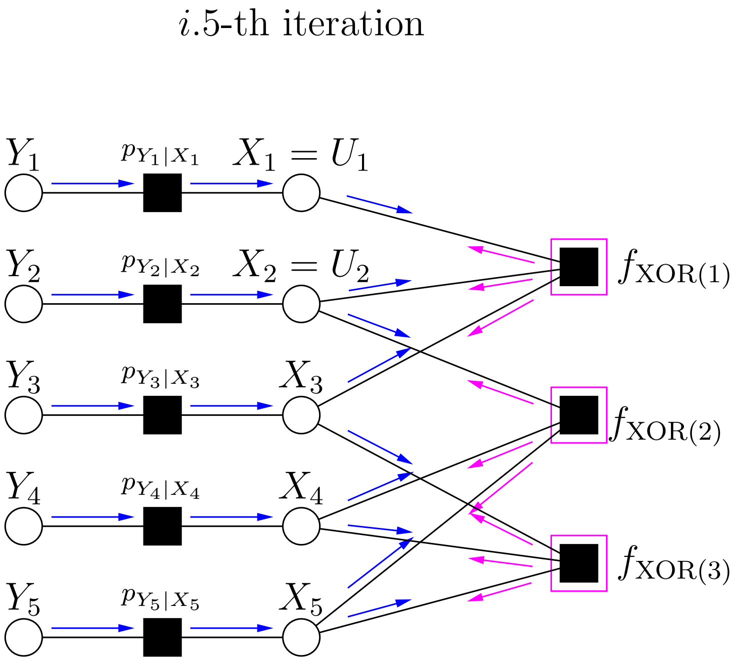
Sum-product algorithm (SPA) decoding:

$$\hat{u}_i^{\text{symbol}}(\mathbf{y}) \approx \underbrace{\underbrace{\operatorname*{argmax}_{u_i \in \mathcal{U}}}_{\text{Decision taking}} \quad \underbrace{\sum_{\substack{\mathbf{u} \in \mathcal{U}^k,\ \mathbf{x} \in \mathcal{X}^n \\ u_i \text{ fixed}}} \underbrace{P_{\mathbf{UXY}}(\mathbf{u}, \mathbf{x}, \mathbf{y}),}_{\text{Joint pmf/pdf}}}_{\substack{\text{Marginal function is approximated} \\ \text{by Sum-Product Algorithm}}}}_{\text{Decision about symbol } u_i \text{ based on symbol-wise decoding}}$$

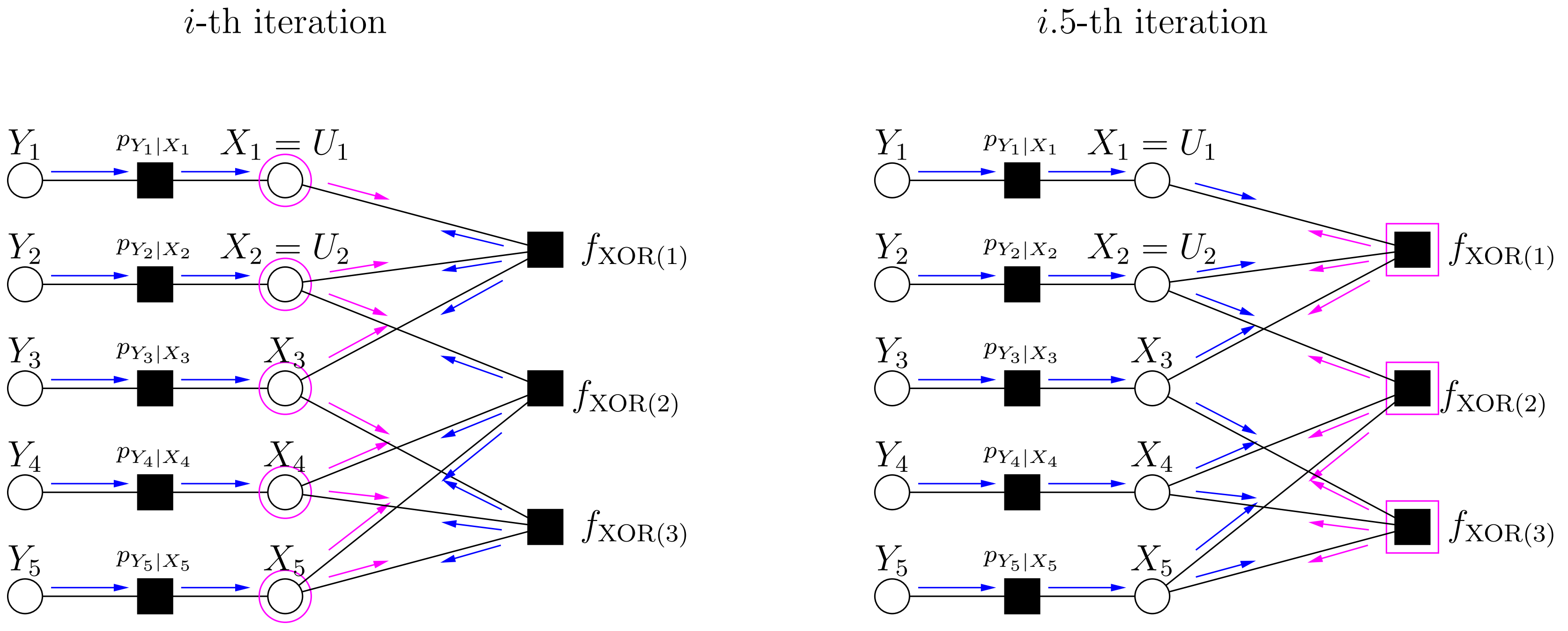
On factor graphs without cycles, the approximation is exact.

SPA Decoding (Factor graph with cycles)





SPA Decoding (Factor graph with cycles)

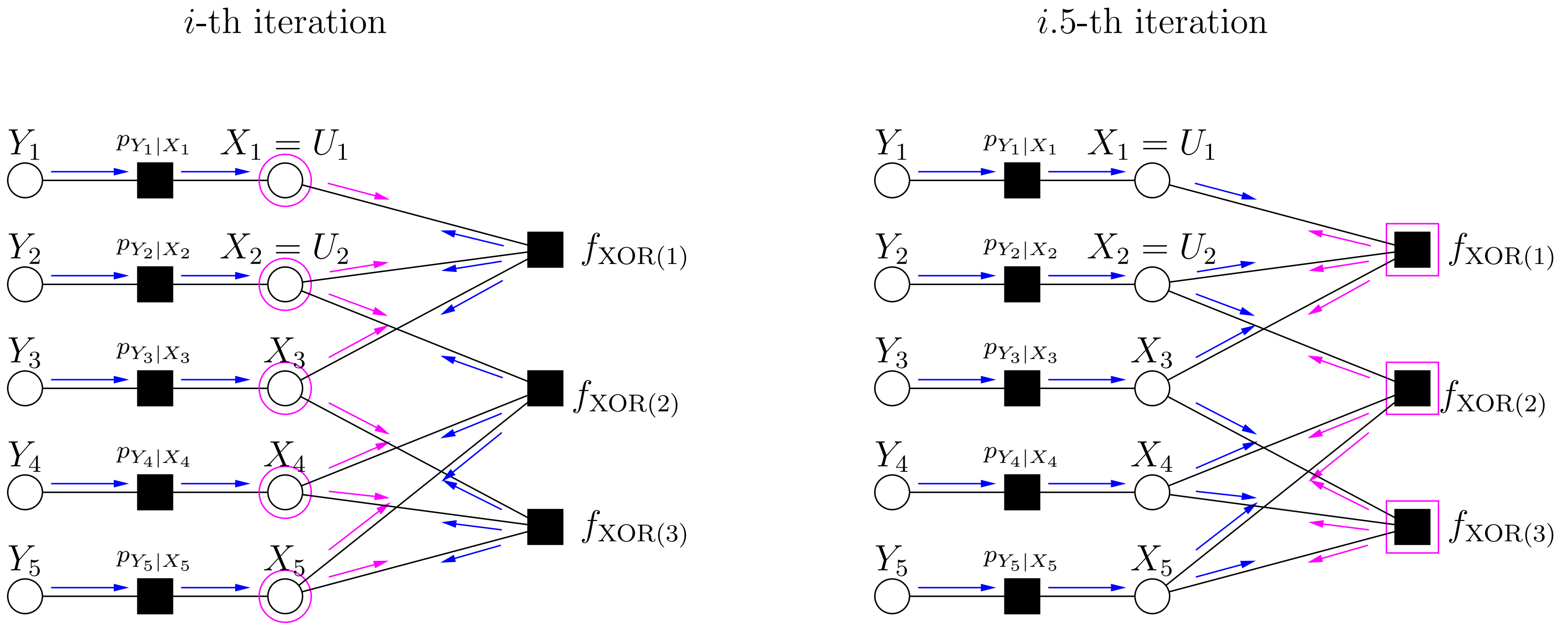

i-th iteration

i.5-th iteration

A message-passing algorithm

- sends messages along the edges,
- does processing of the messages at the vertices.

SPA Decoding (Factor graph with cycles)

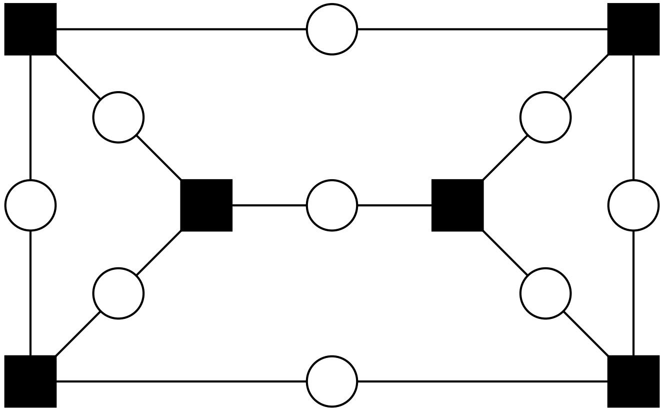


A message-passing algorithm

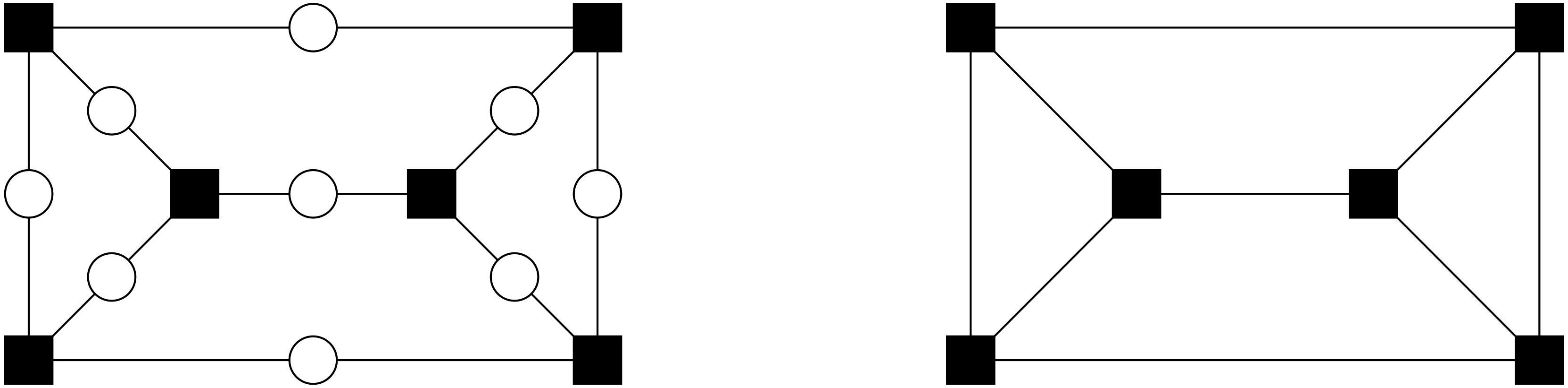
- sends messages along the edges,
- does processing of the messages at the vertices.

Note: all operations are performed locally!

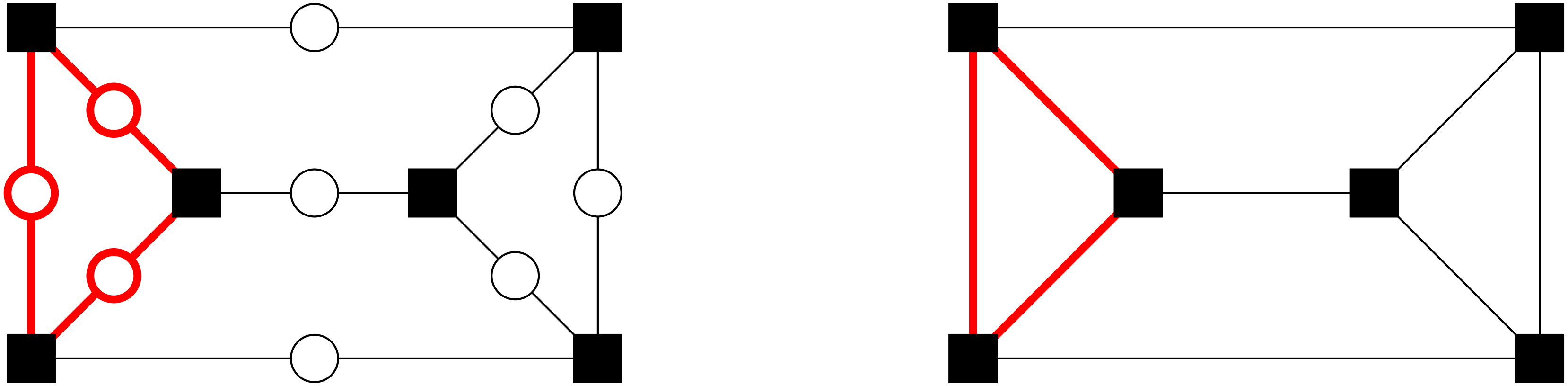
Factor Graph of a Cycle Code



Factor Graph of a Cycle Code

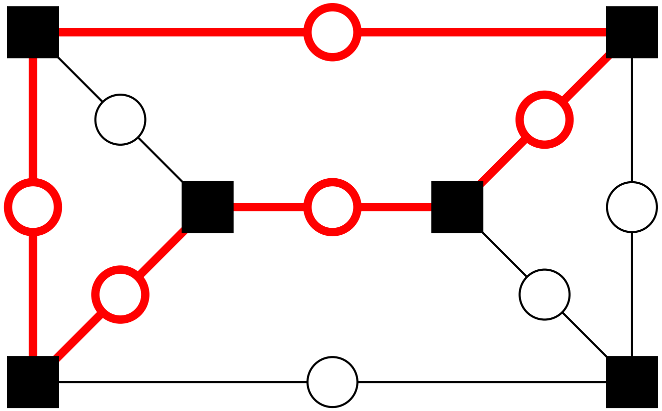


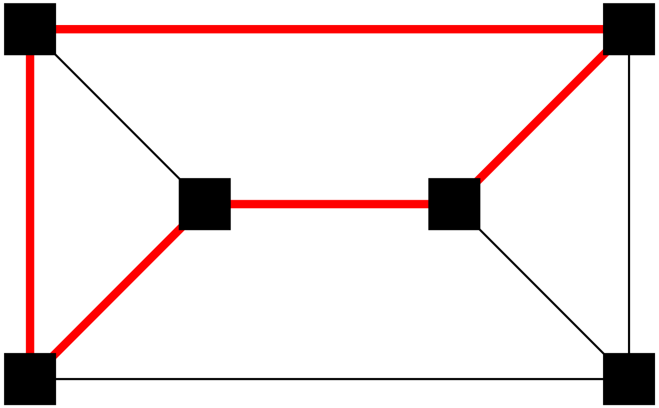
Factor Graph of a Cycle Code



Cycle codes are called cycle codes because codewords correspond to simple cycles (or to the symmetric difference set of simple cycles) in the Tanner/factor graph.

Factor Graph of a Cycle Code





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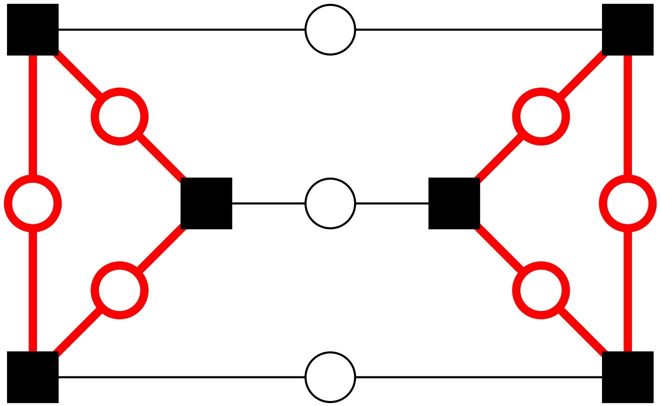
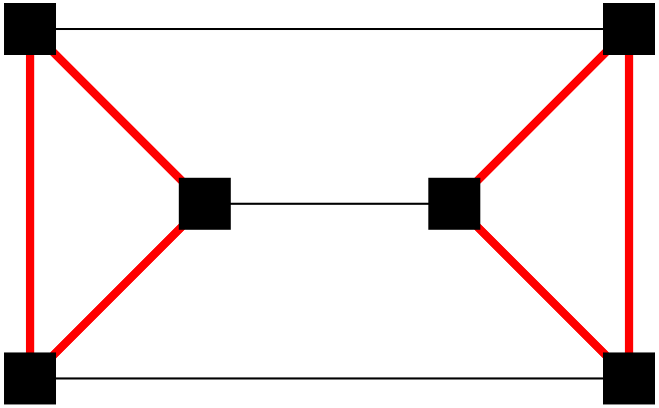
Factor Graph of a Cycle Code





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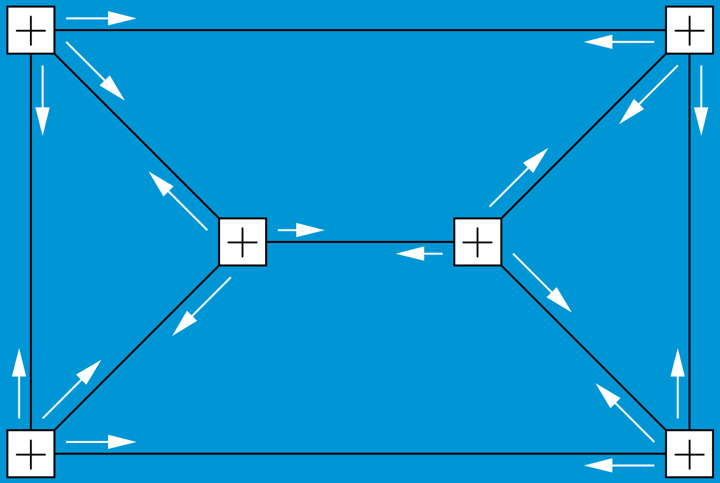
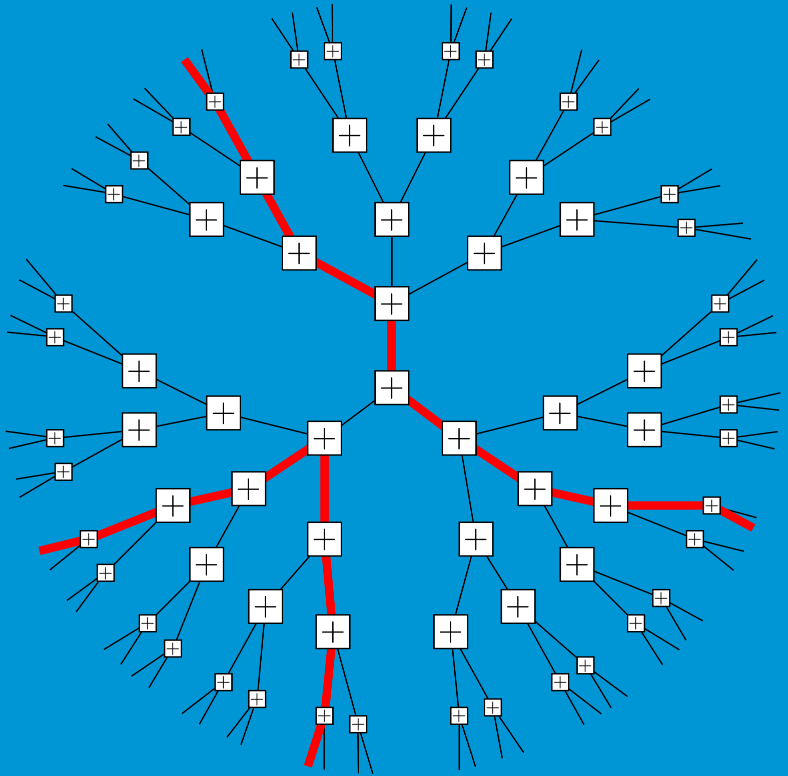
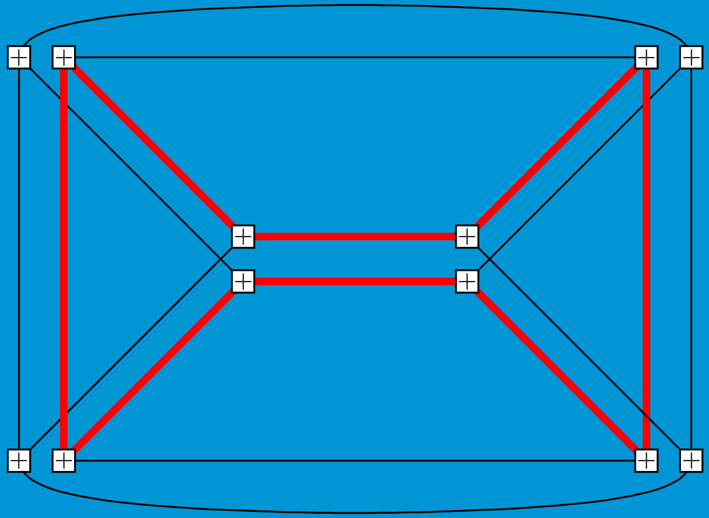
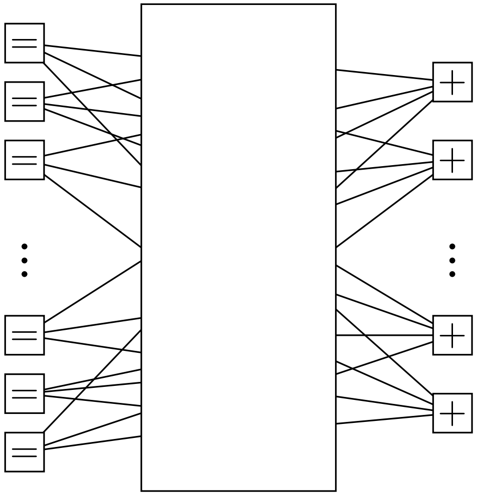




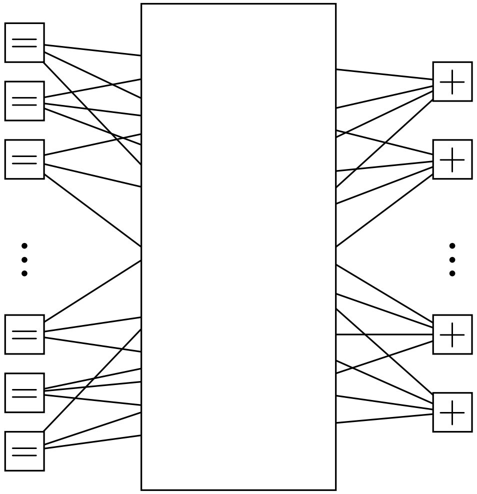


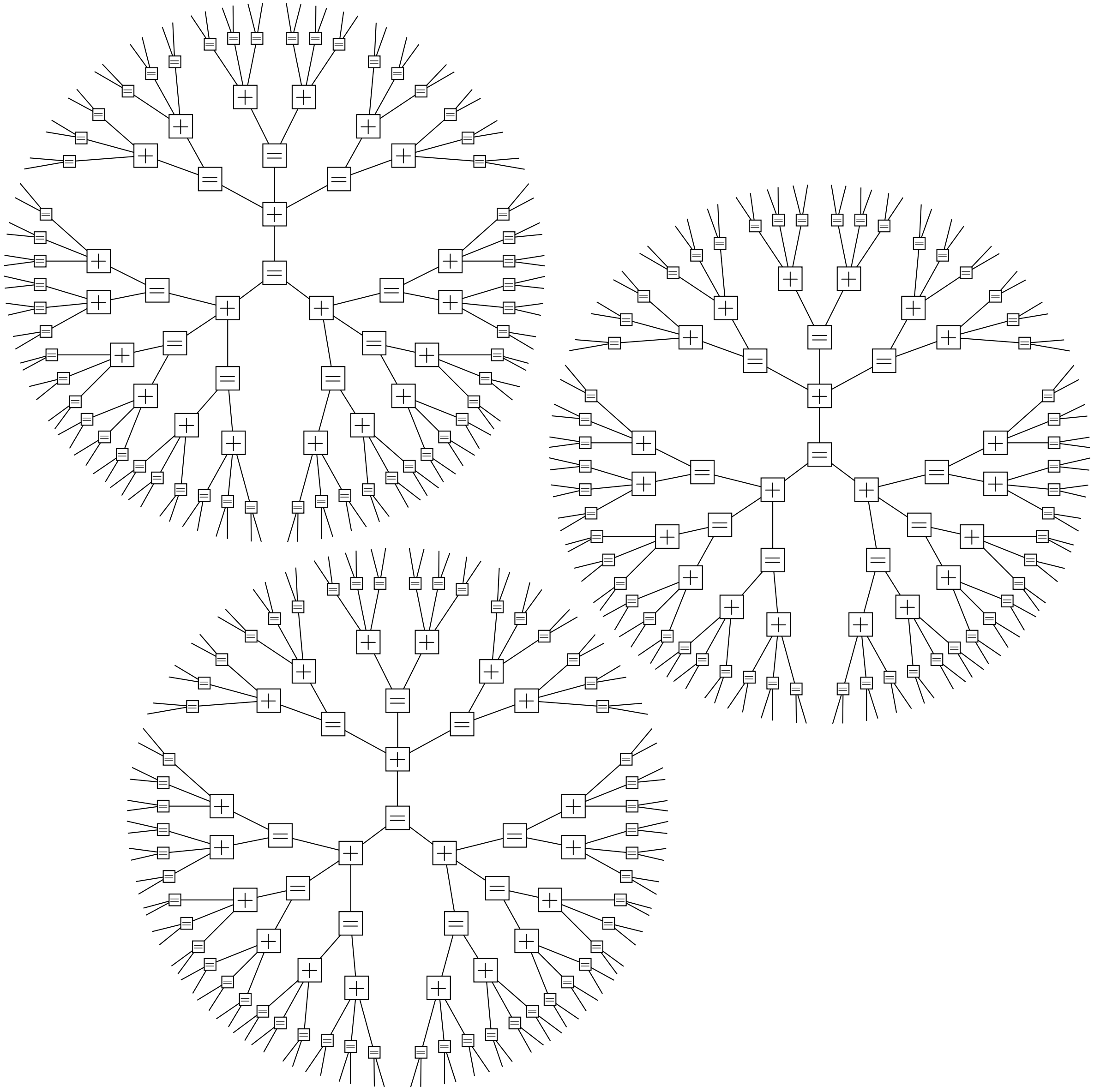
$$\zeta(\mathsf{V}_1,\ldots,\mathsf{V}_n) \;=\; \sum_{\mathbf{k}} \zeta_{\mathbf{k}} \mathbf{V}^{\mathbf{k}} \;=\; \prod_{[\Gamma]} \frac{1}{1-g(\Gamma,\mathbf{V})}$$

Computation trees

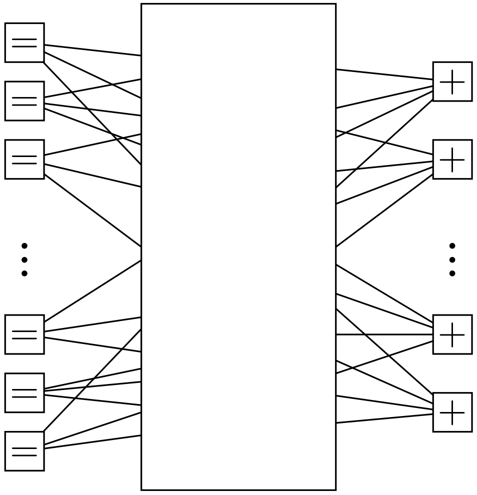


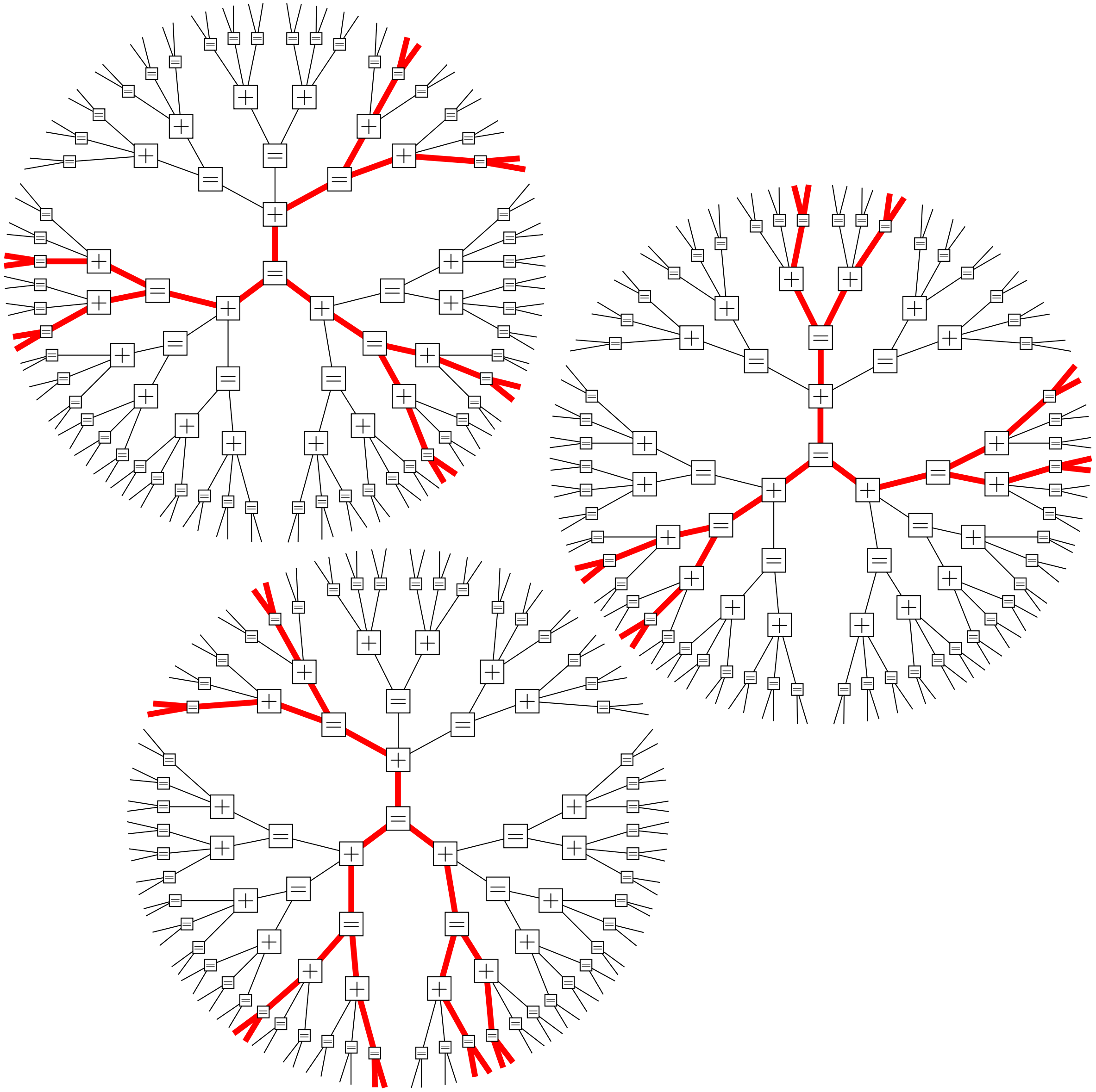
Computation trees

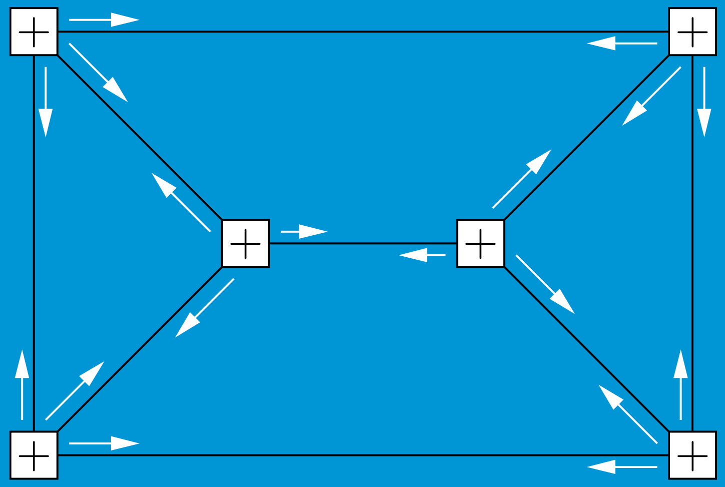


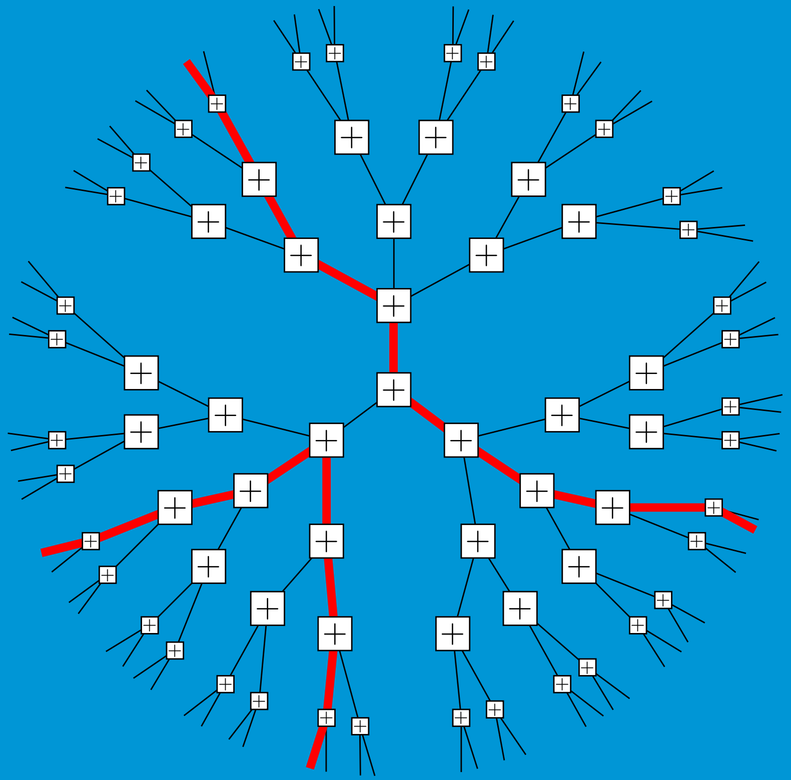


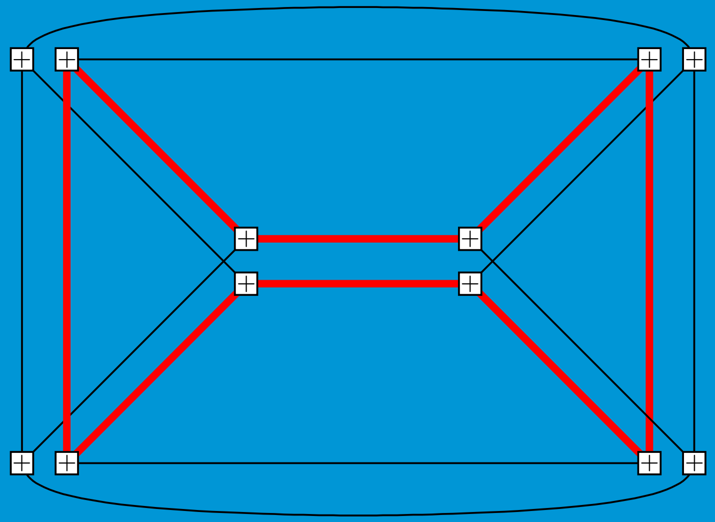
Computation trees











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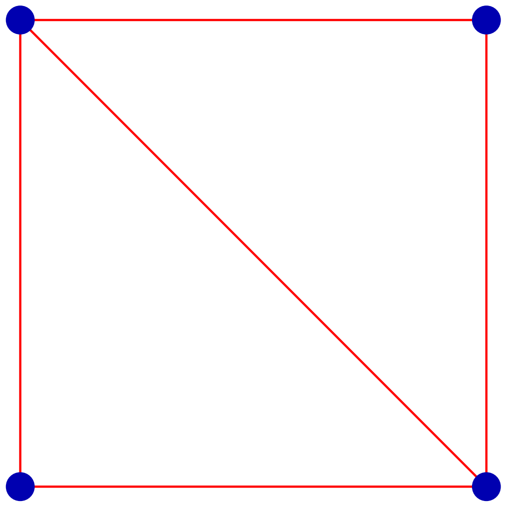
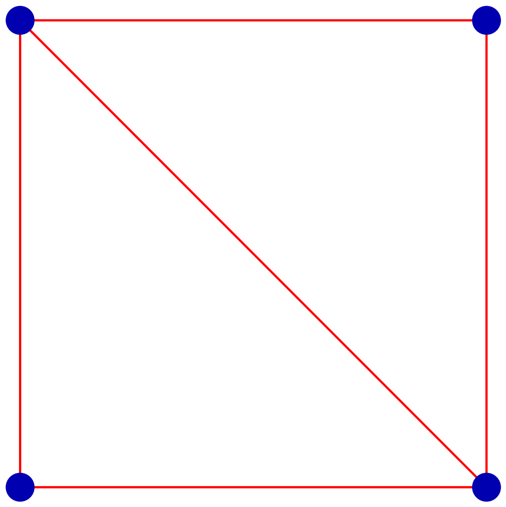
Finite Graph Covers



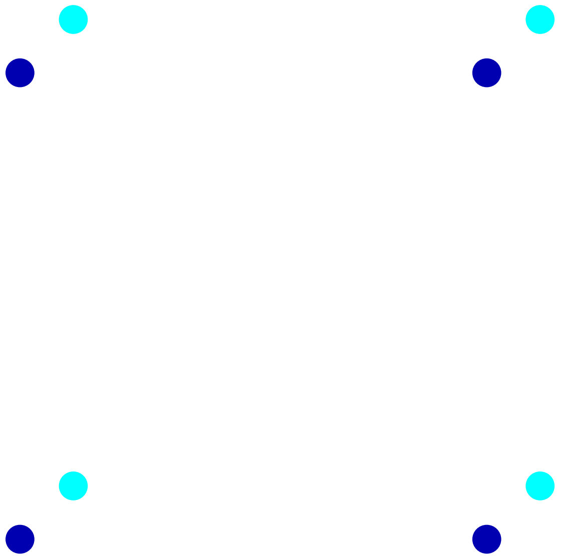
original graph

Definition: A double cover of a graph is . . .

Finite Graph Covers



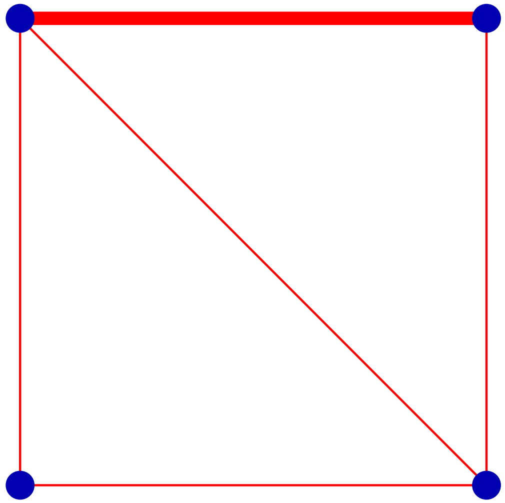
original graph



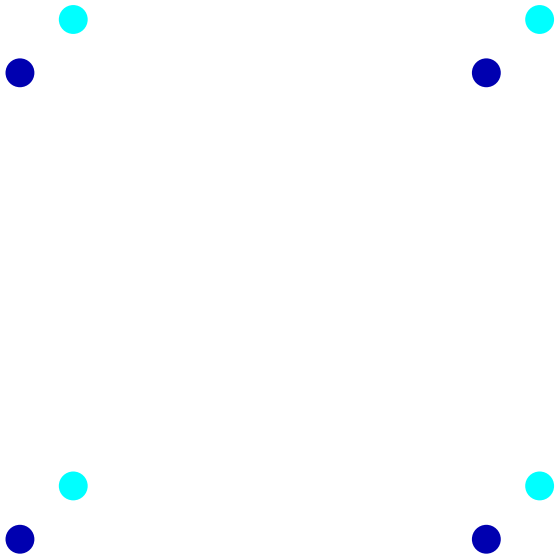
2-fold cover of original graph

Definition: A double cover of a graph is . . .

Finite Graph Covers



original graph



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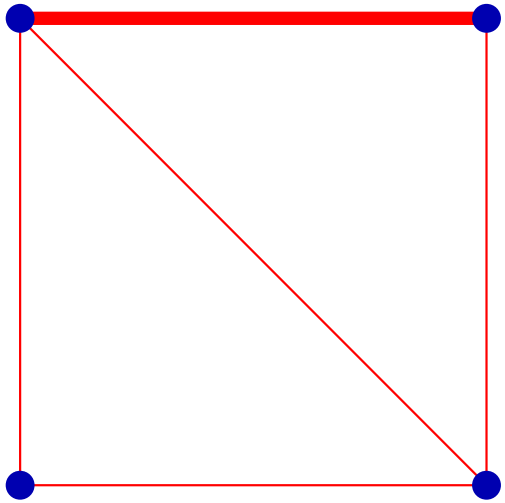
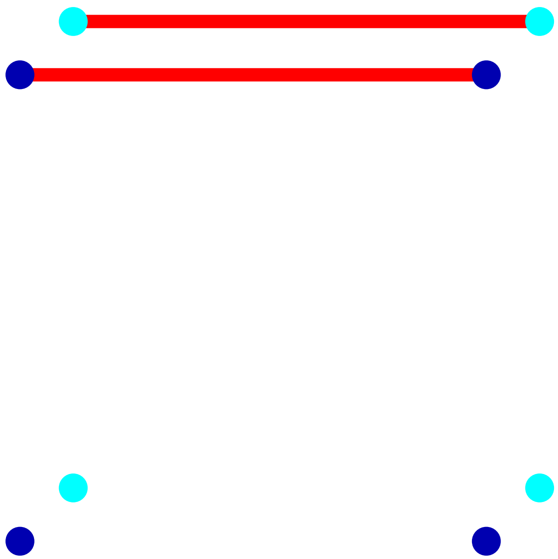


original graph



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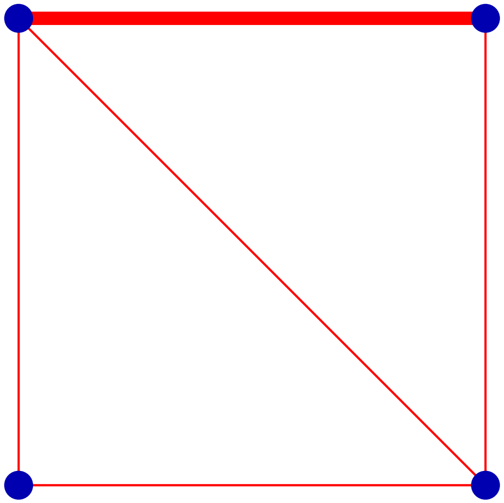
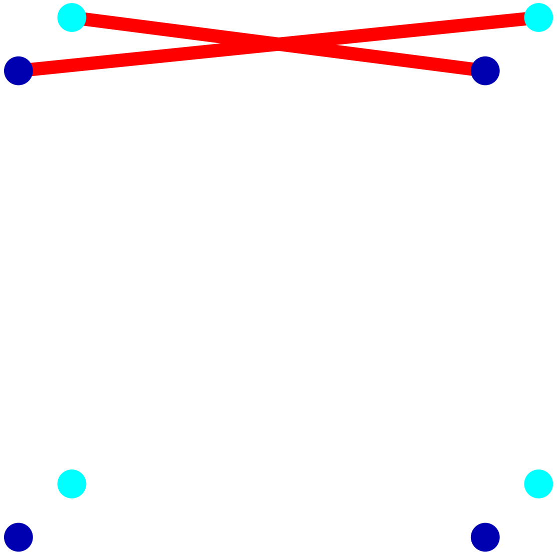


original graph



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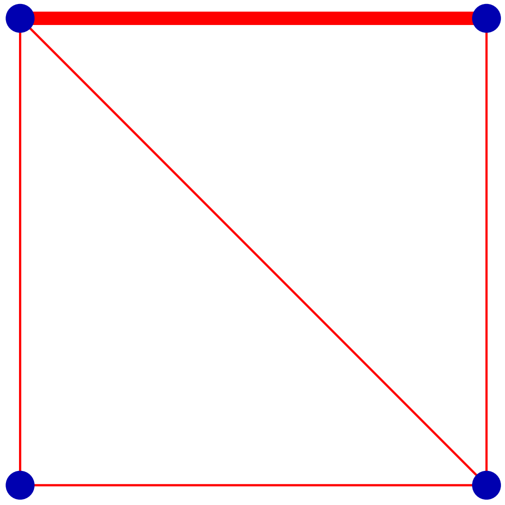
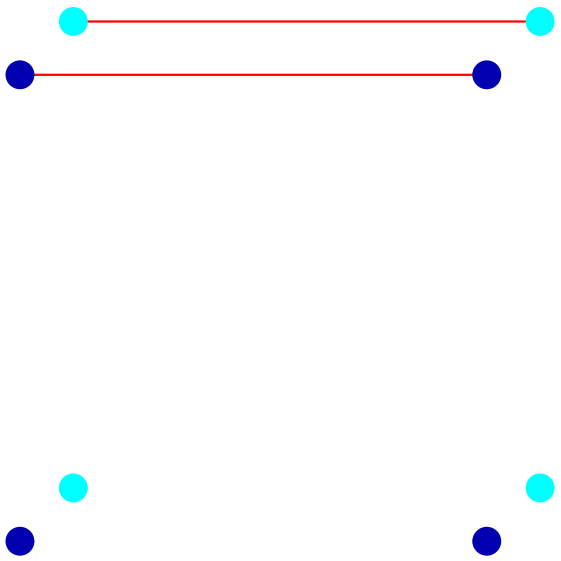


original graph



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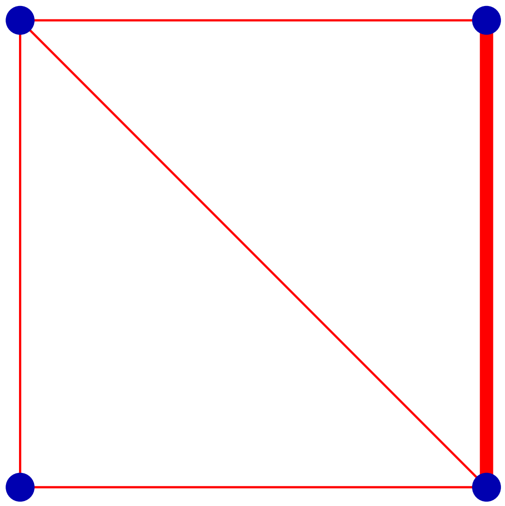
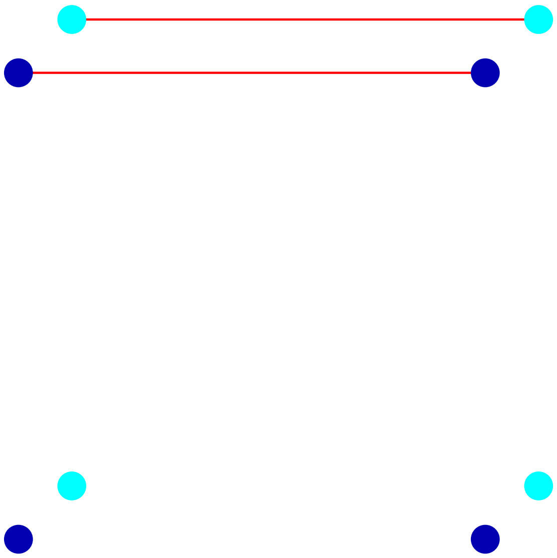


original graph



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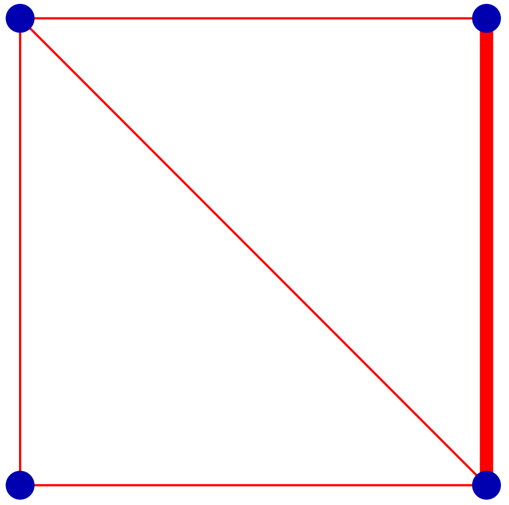

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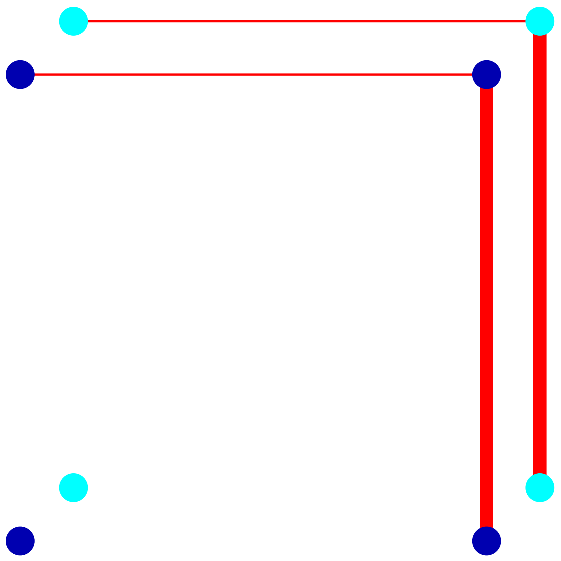

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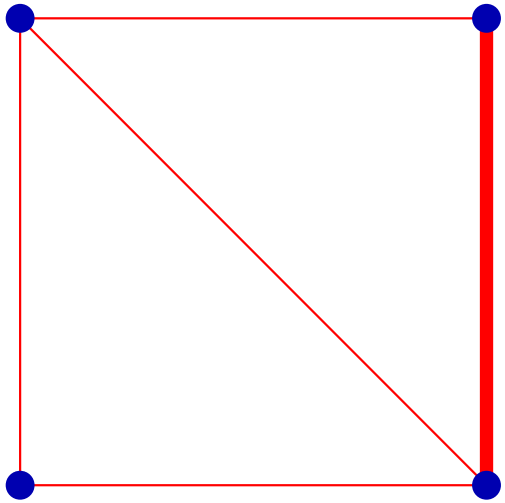

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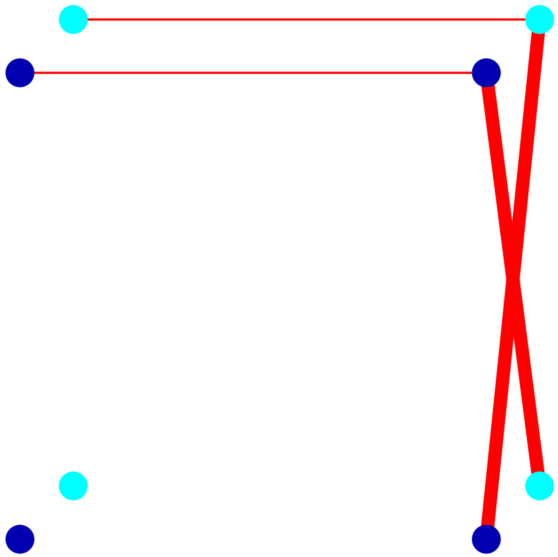

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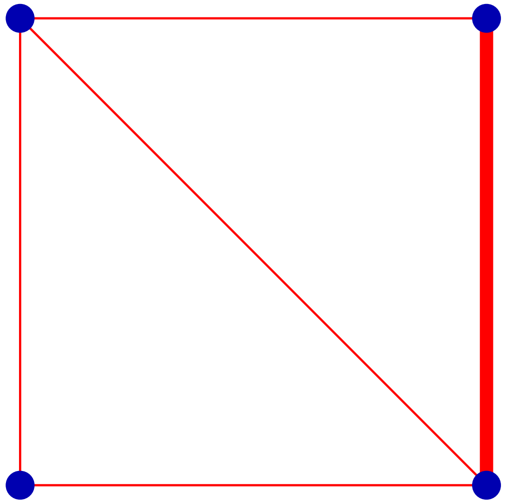

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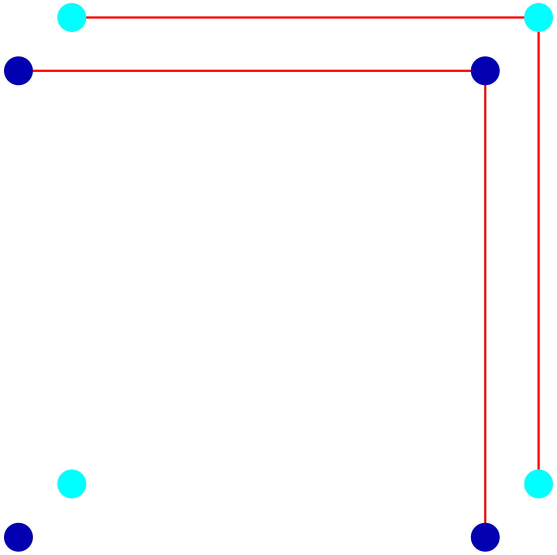

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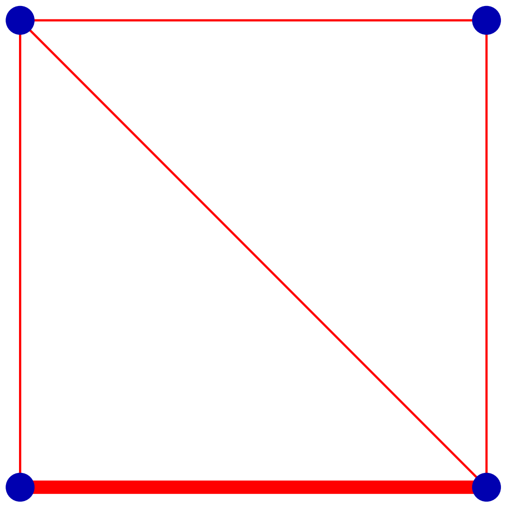
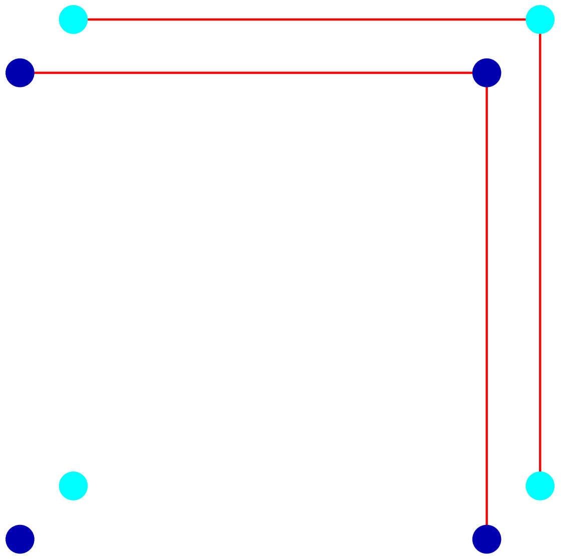


original graph



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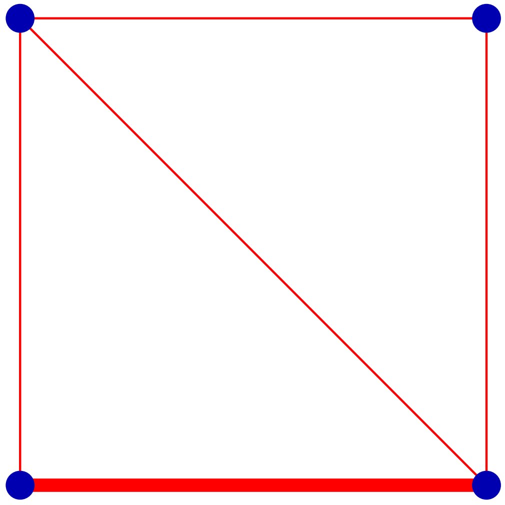
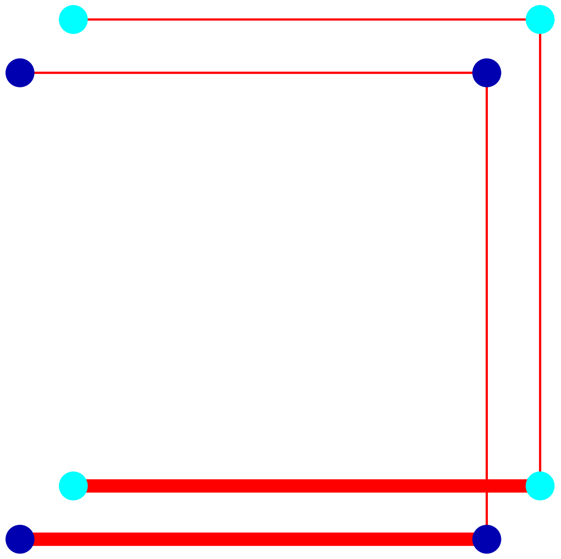
Finite Graph Covers



original graph



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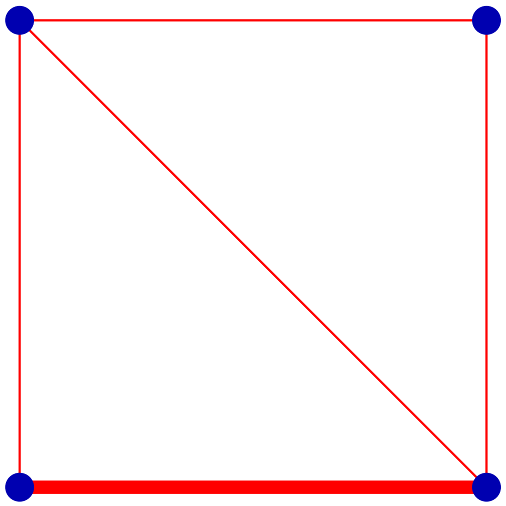
Finite Graph Covers



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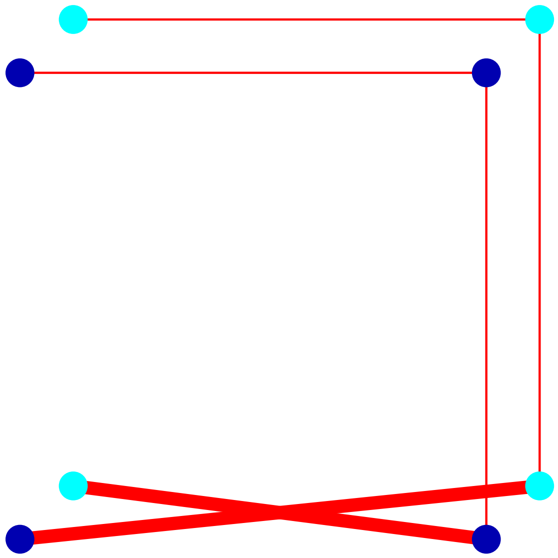

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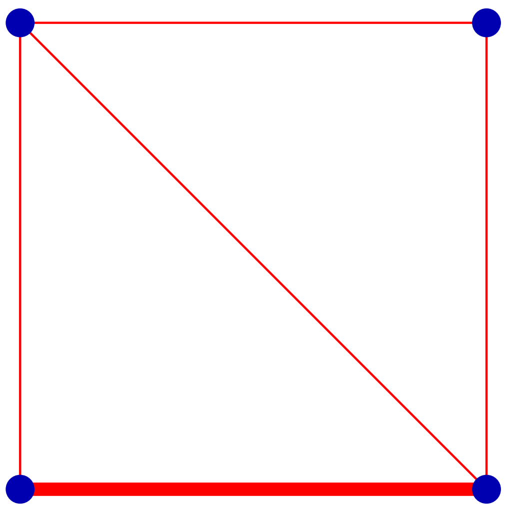
Finite Graph Covers



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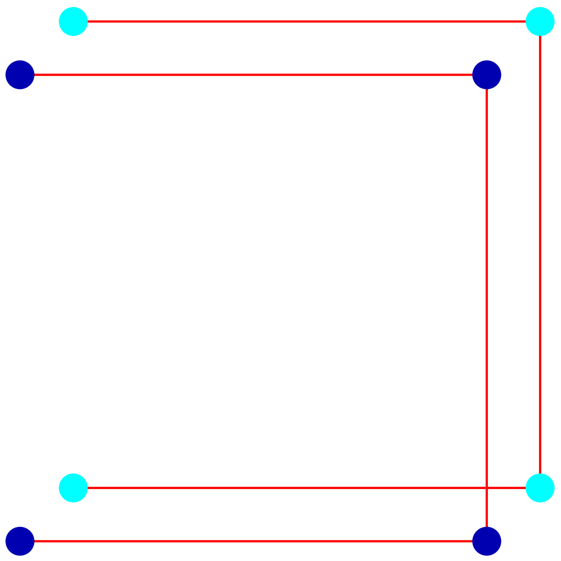

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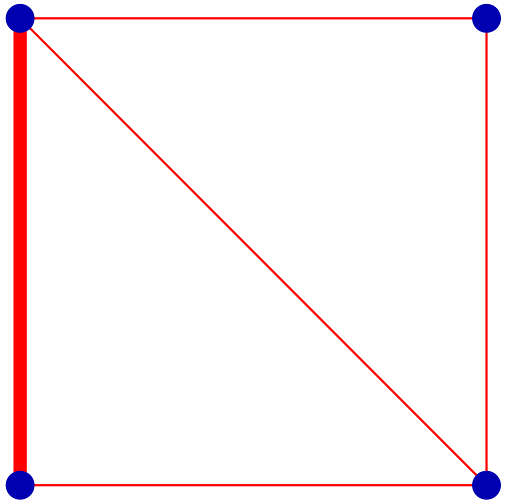

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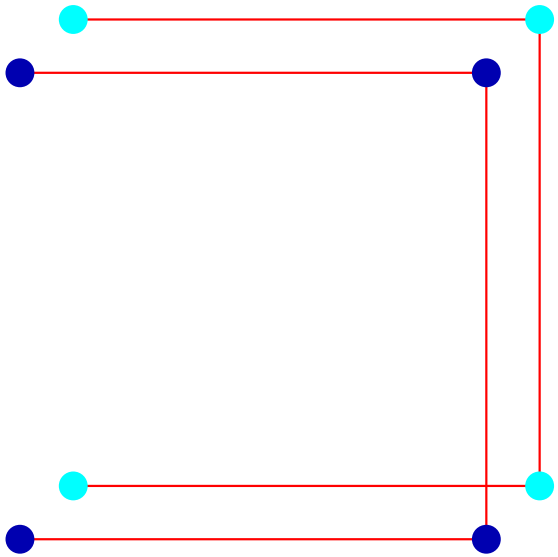

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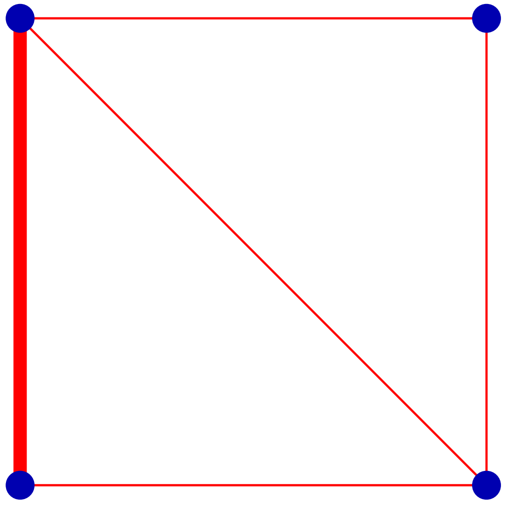

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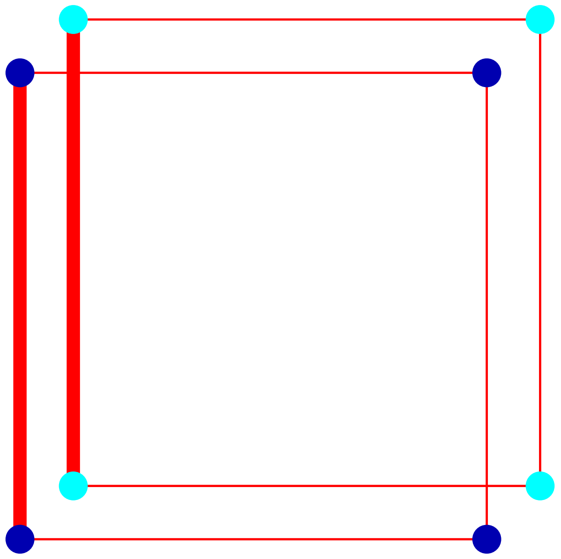

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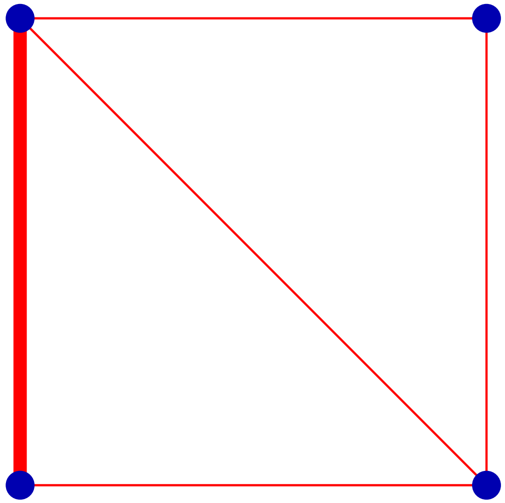
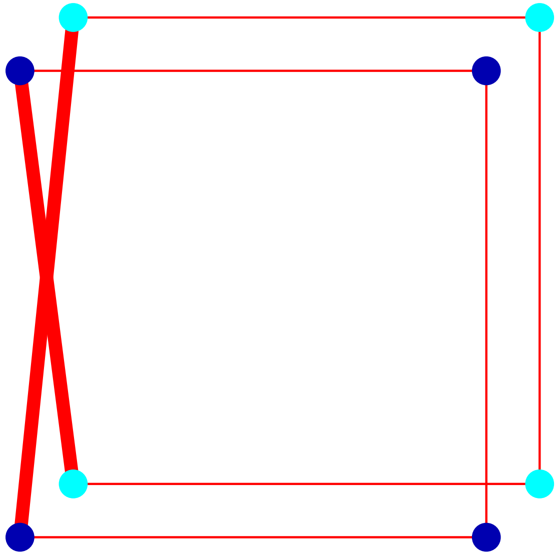


original graph



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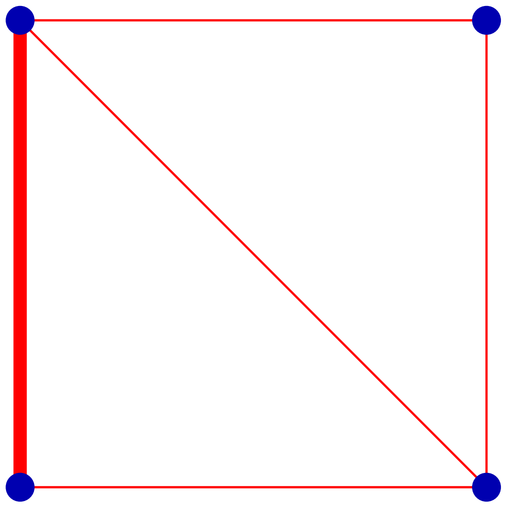

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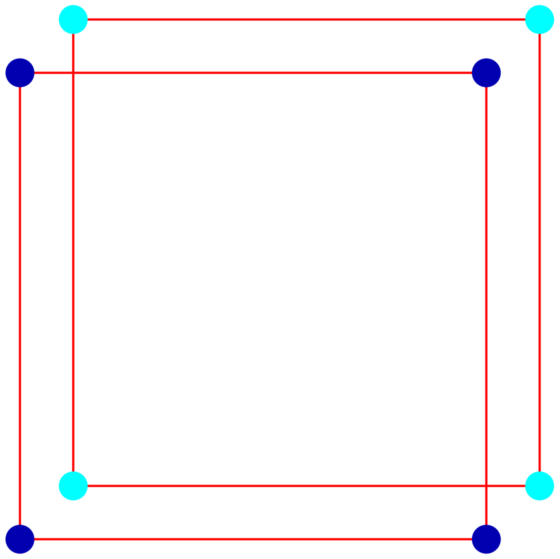

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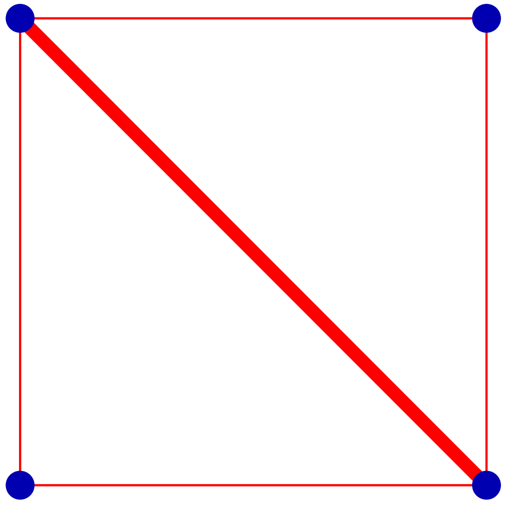
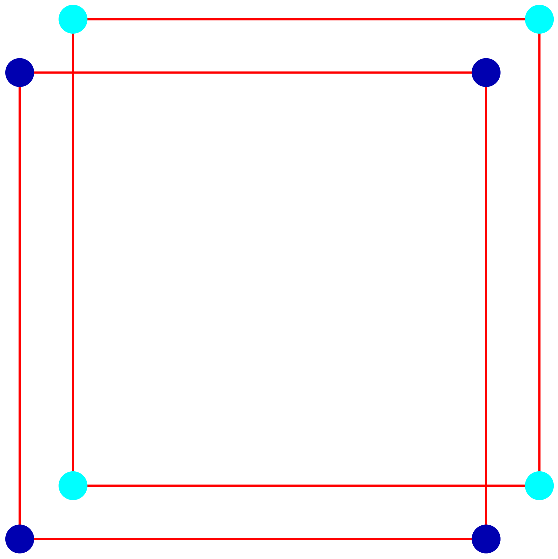
Finite Graph Covers



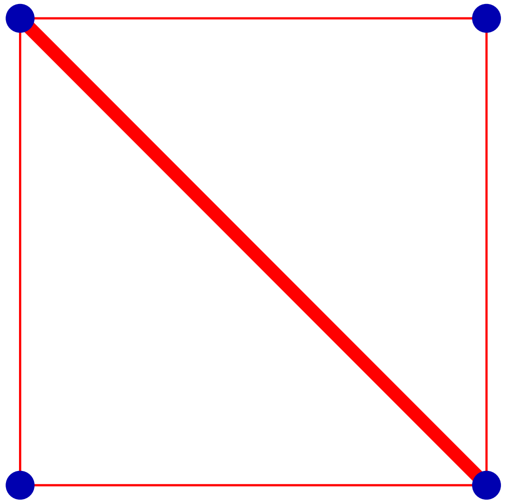
original graph



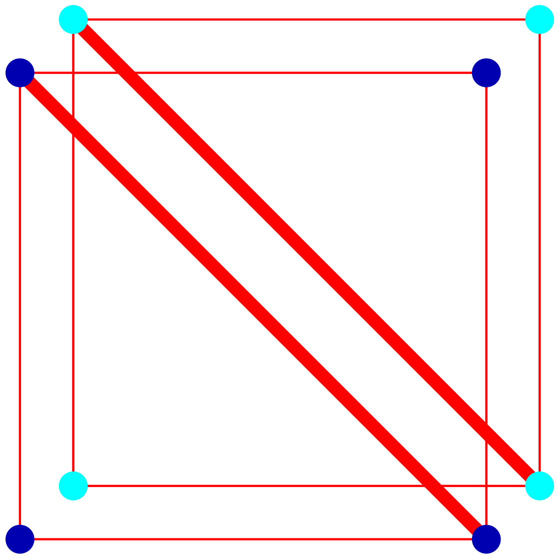
2-fold cover of original graph

Definition: A double cover of a graph is . . .

Finite Graph Covers



original graph



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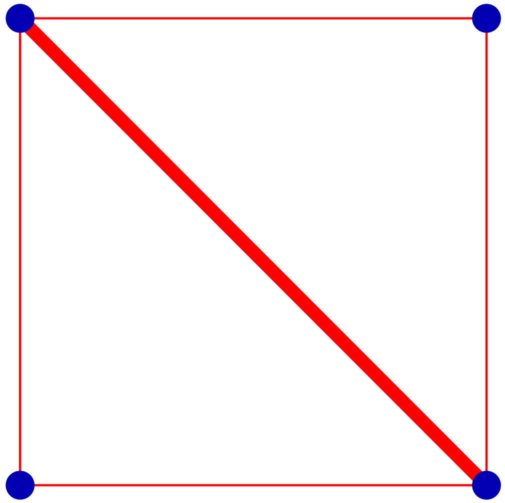
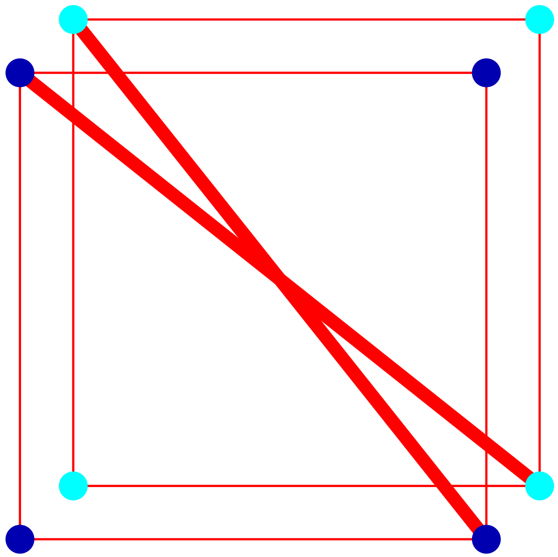


original graph



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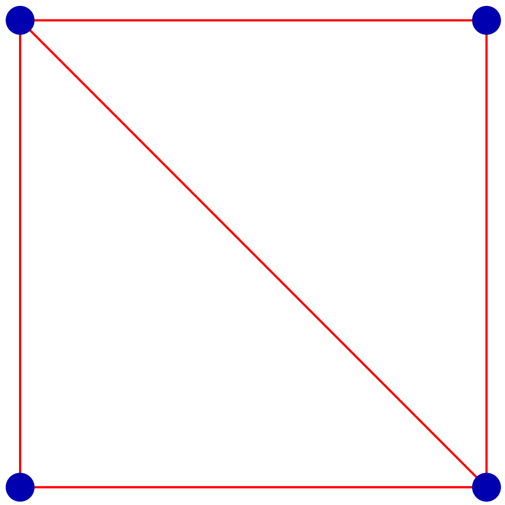
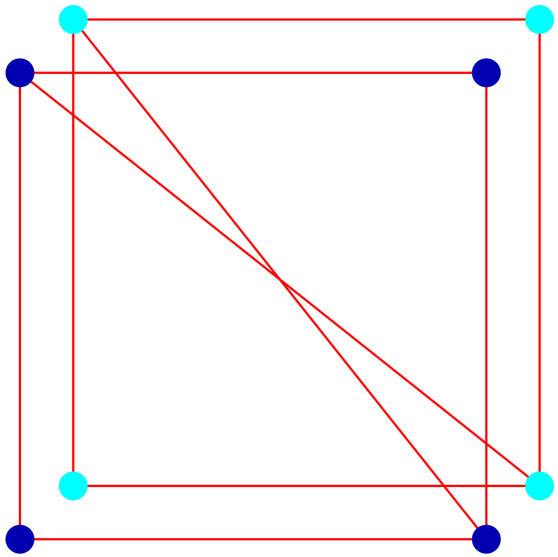


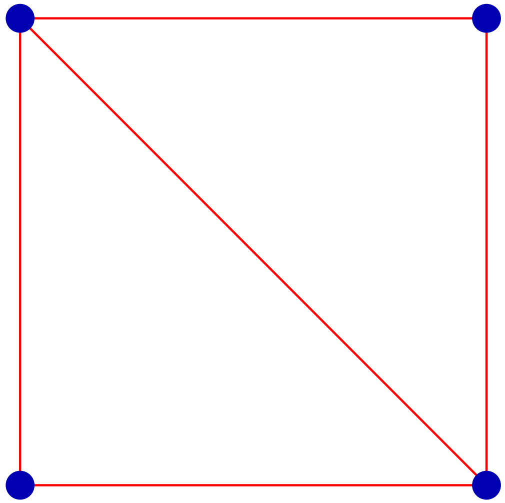
original graph



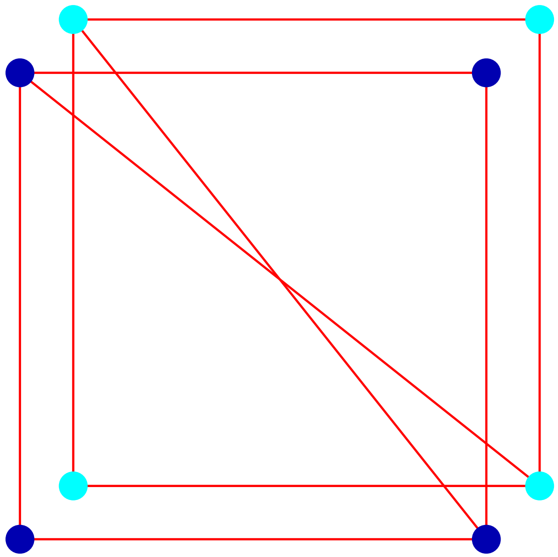
2-fold cover of original graph

Definition: A double cover of a graph is . . .

Finite Graph Covers



original graph

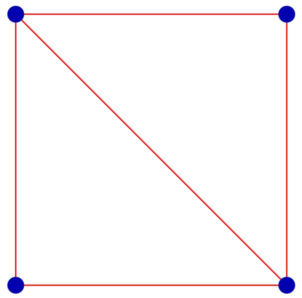


2-fold cover of original graph

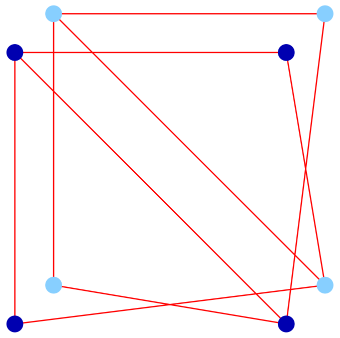
Definition: A double cover of a graph is . . .

Note: the above graph has $2! \cdot 2! \cdot 2! \cdot 2! \cdot 2! = (2!)^5$ double covers.

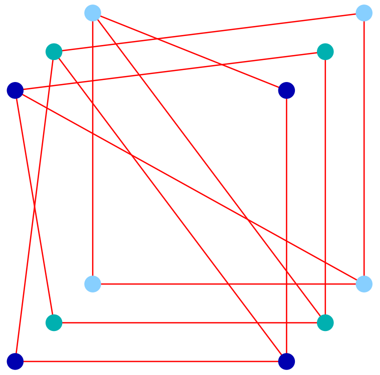
Graph Covers



original graph



(a possible)
double cover of
the original graph

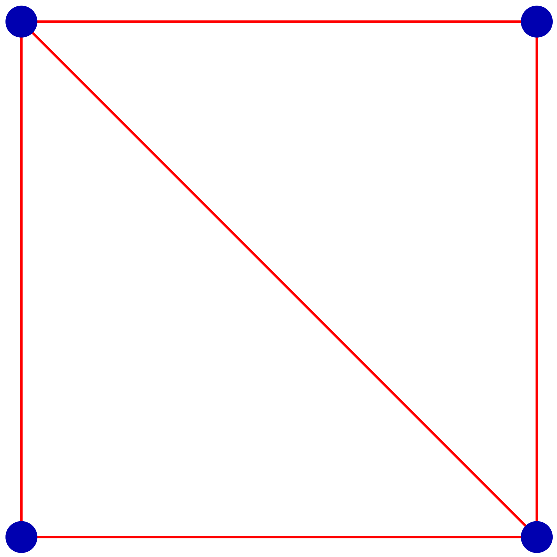


(a possible)
triple cover of
the original graph

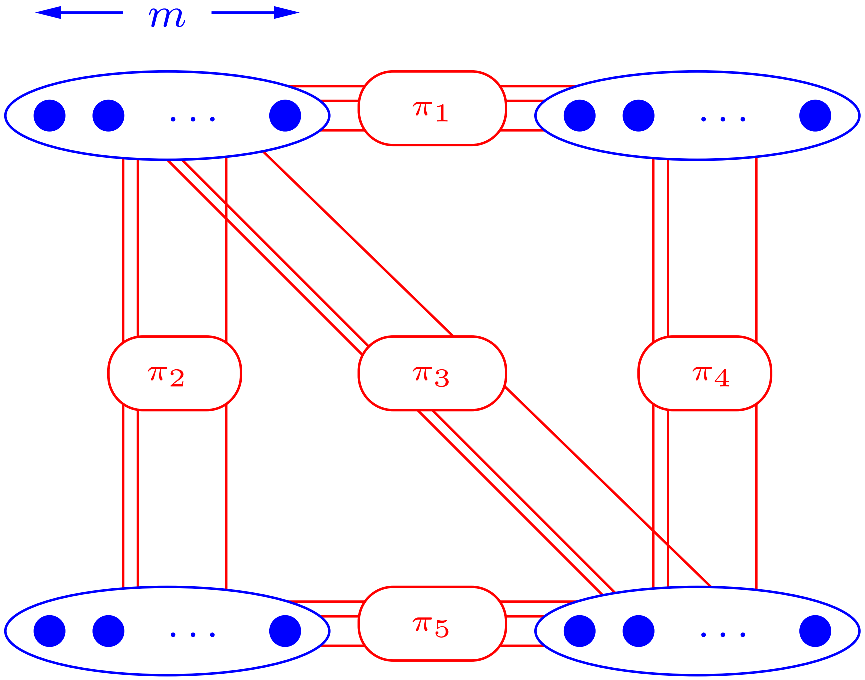
. . .

Besides double covers, a graph also has many triple covers, quadruple covers, quintuple covers, etc.

Graph Covers



original graph



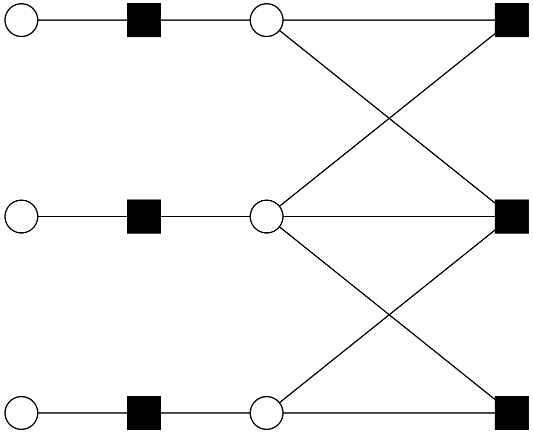
(possible)
m-fold cover of
original graph

An m-fold cover is also called a cover of degree m.

Do not confuse this degree with the degree of a vertex!

Graph Covers and the Sum-Product Algorithm

Consider this factor graph:



Graph Covers and the Sum-Product Algorithm

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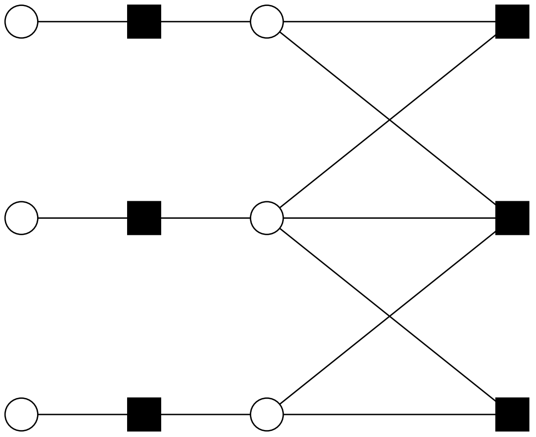
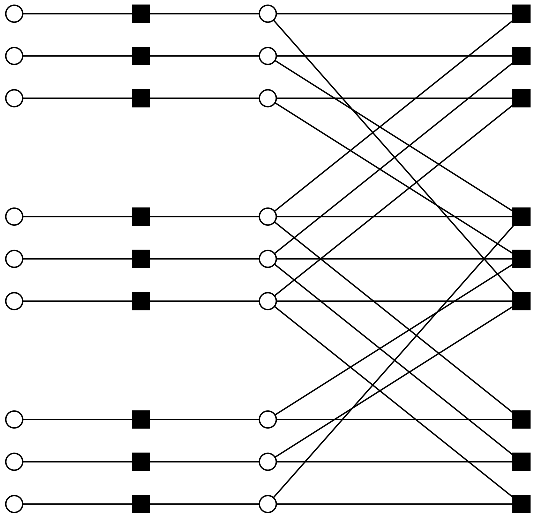
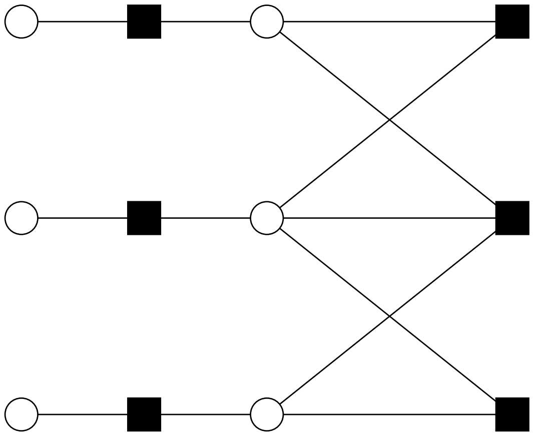


Here is a so-called triple cover of the above factor graph:



Graph Covers and the Sum-Product Algorithm

Consider this factor graph:



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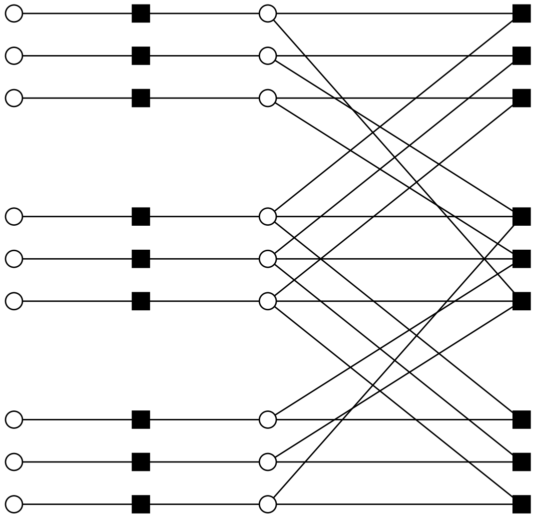
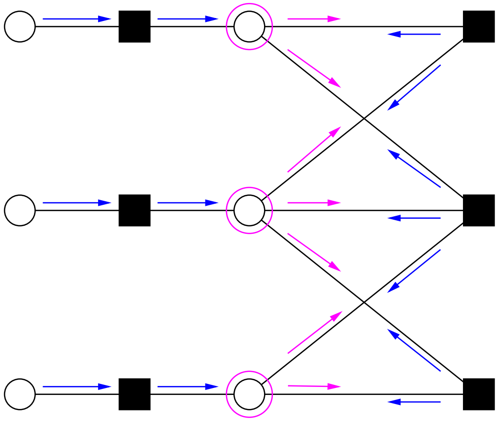


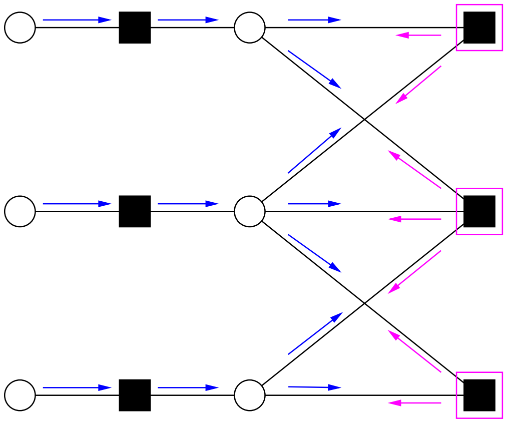
Why do factor graph covers matter?

Graph Covers and the Sum-Product Algorithm

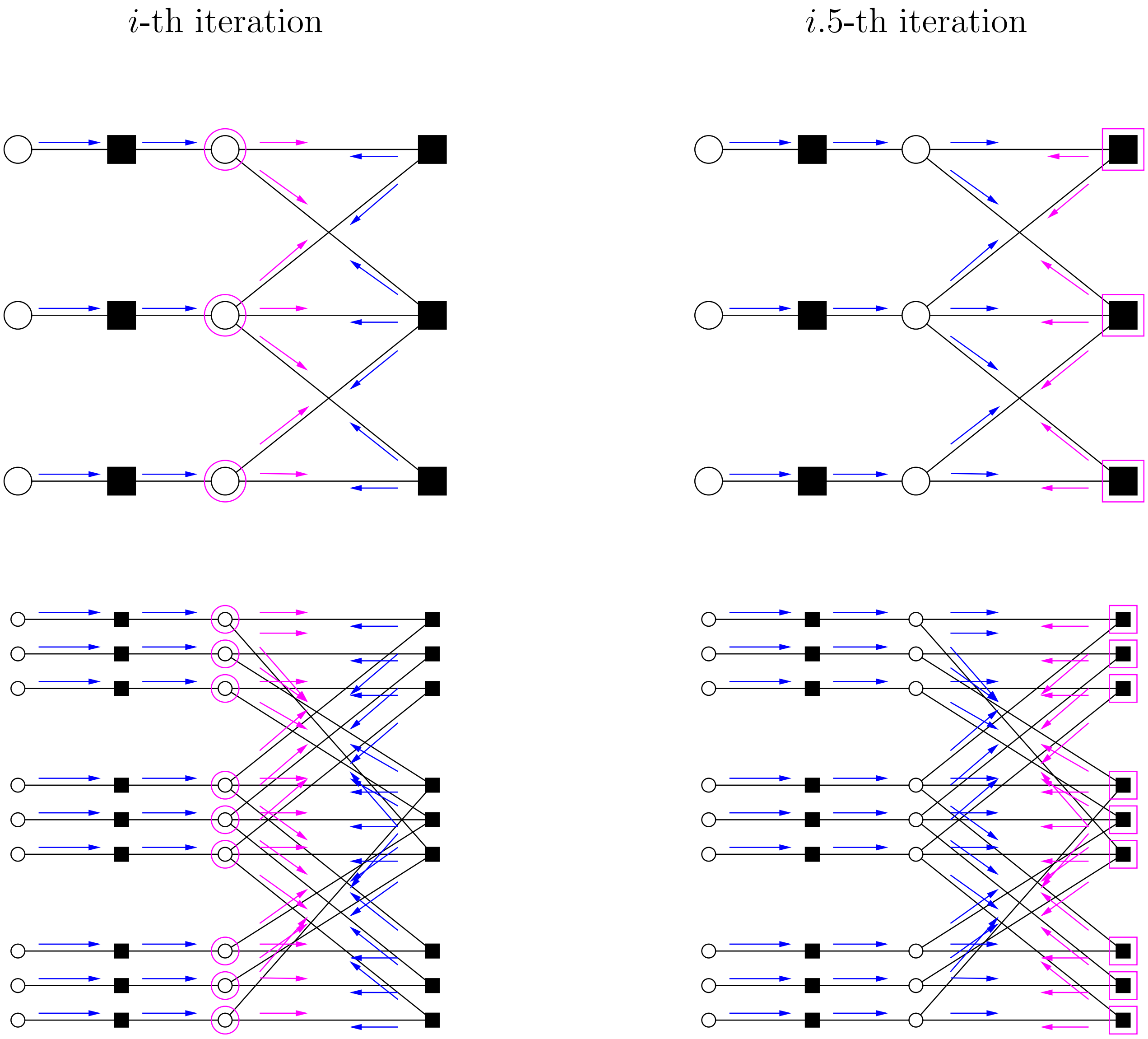
i-th iteration



$i.5$-th iteration



Graph Covers and the Sum-Product Algorithm

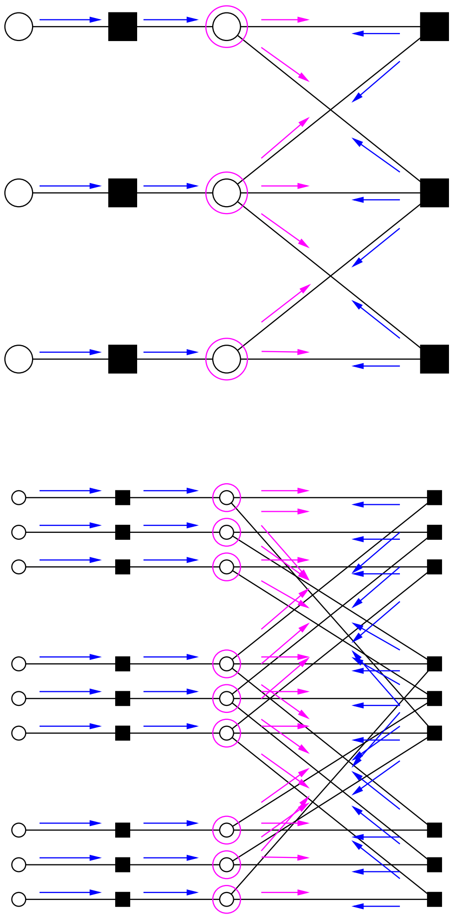


i-th iteration

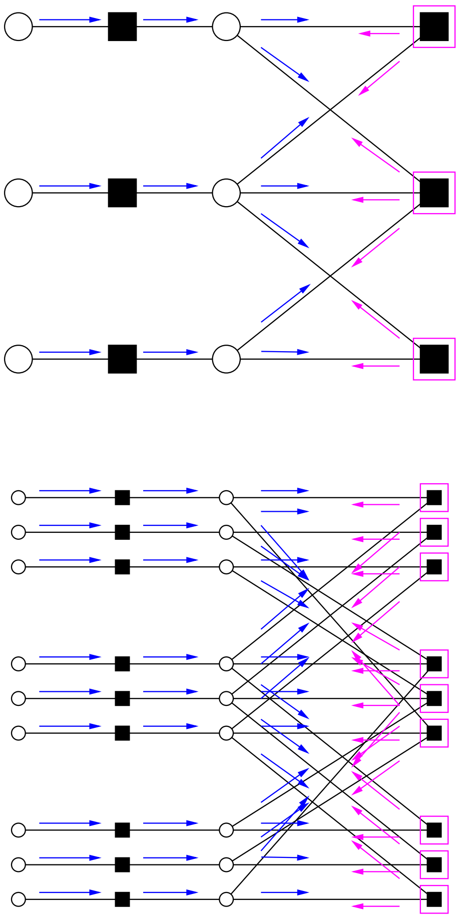
$i.5$-th iteration

Graph Covers and the Sum-Product Algorithm

i-th iteration

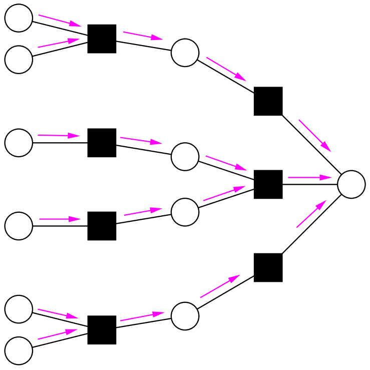


$i.5$-th iteration



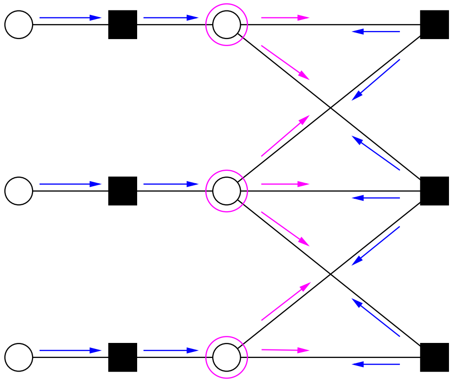
computation tree (without channel function nodes) . . .

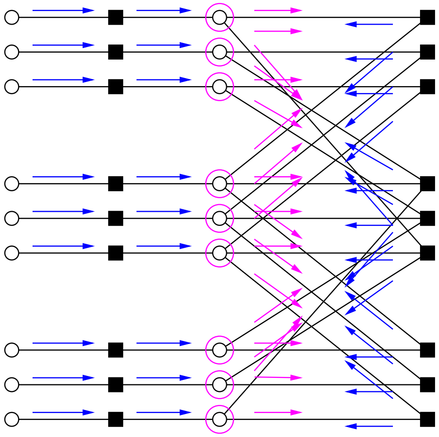
. . . where root is bit node 2



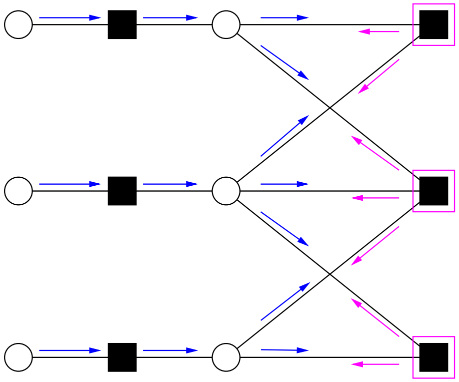
Graph Covers and the Sum-Product Algorithm

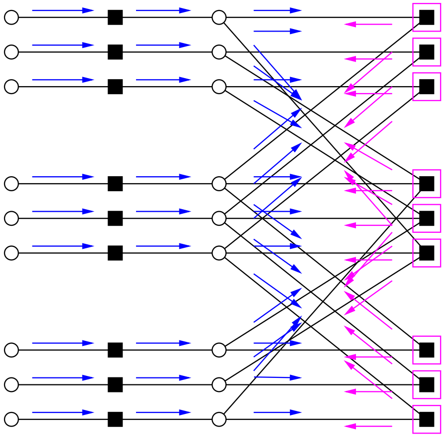
i-th iteration





$i.5$-th iteration





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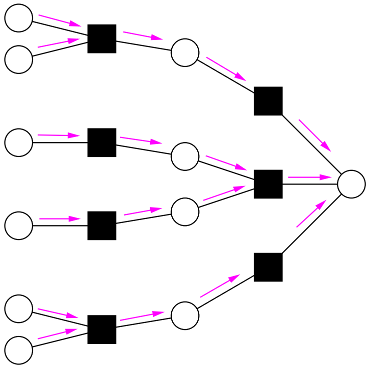
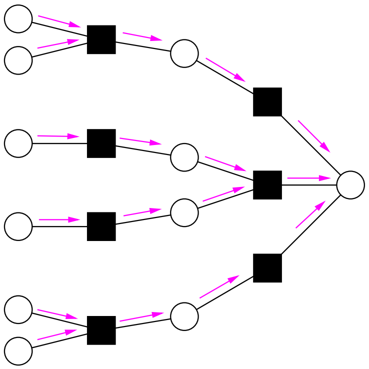
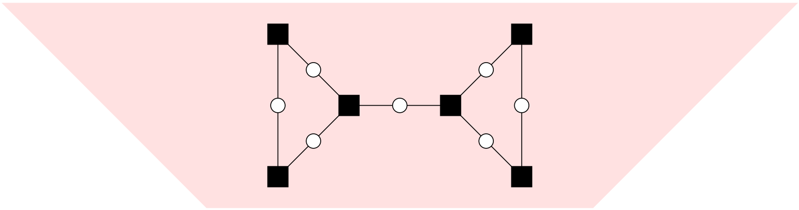


. . . where root is a copy of bit node 2



Graph Cover Hierarchy

Graph Cover Hierarchy



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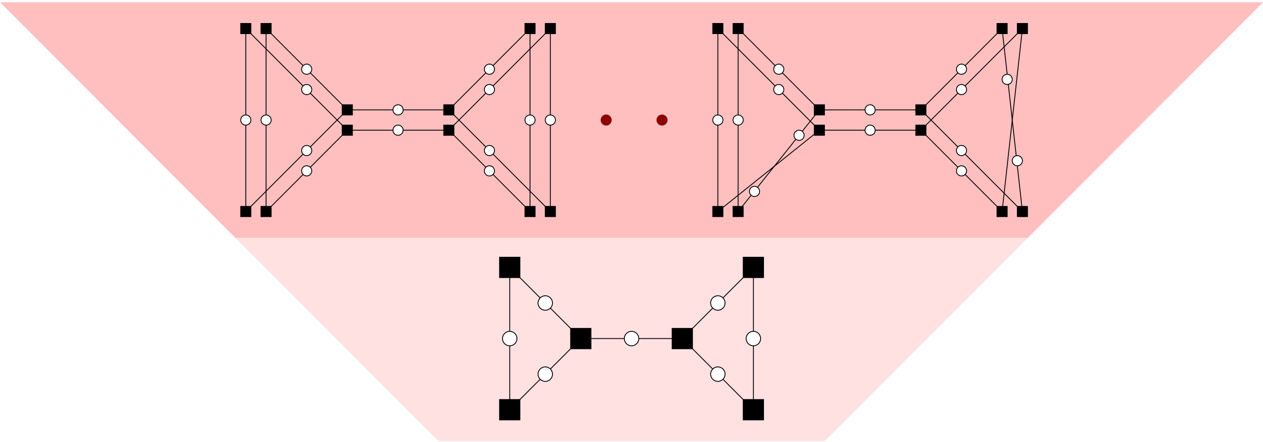
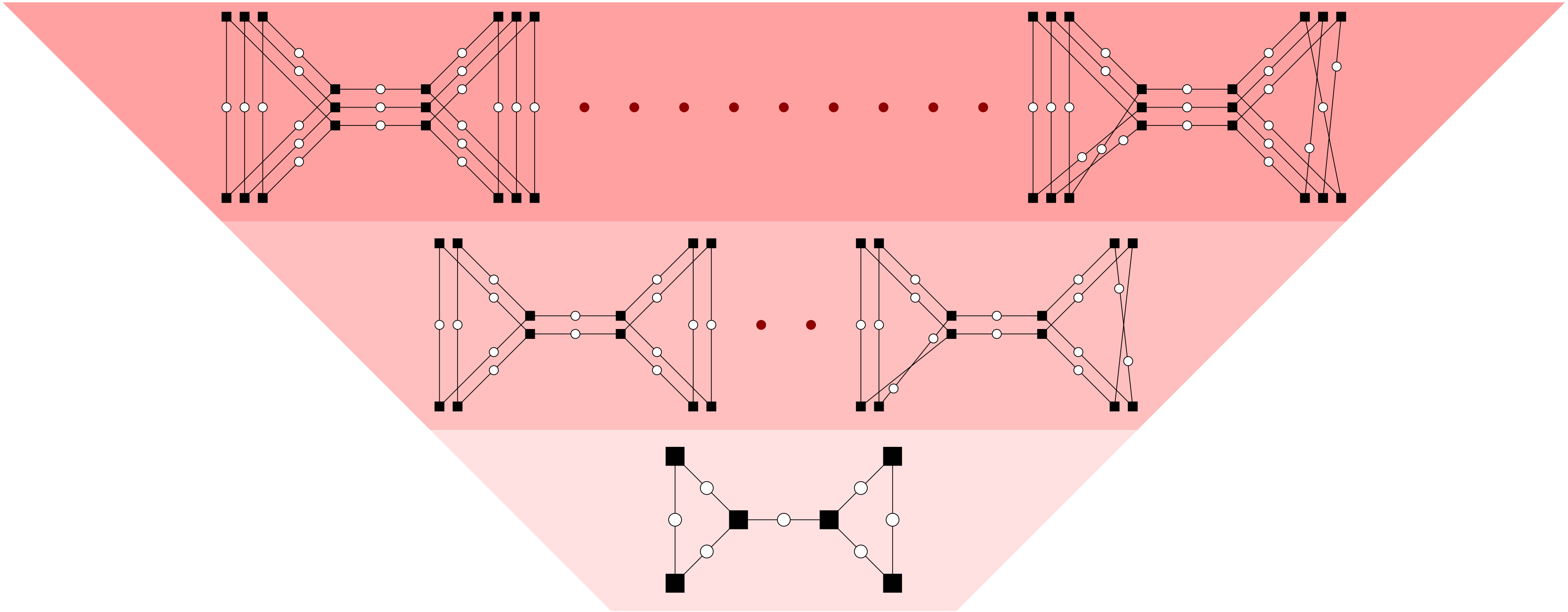
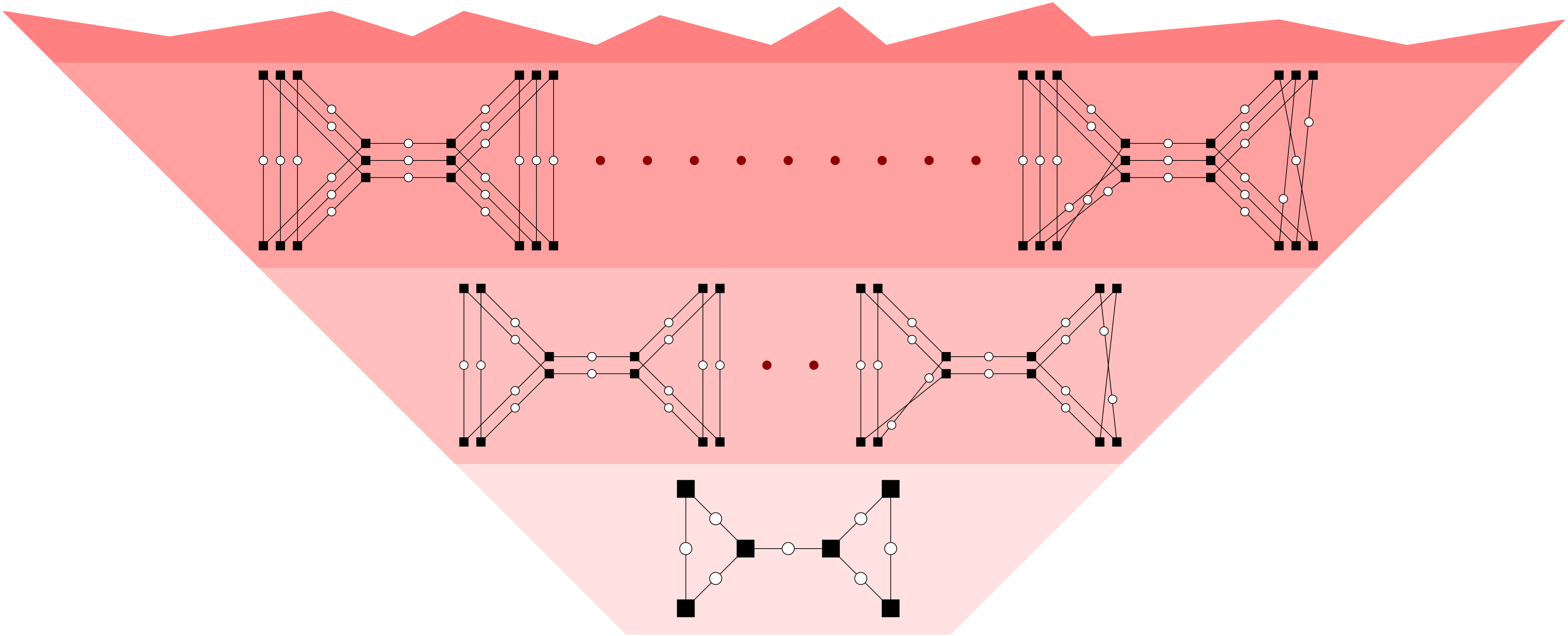
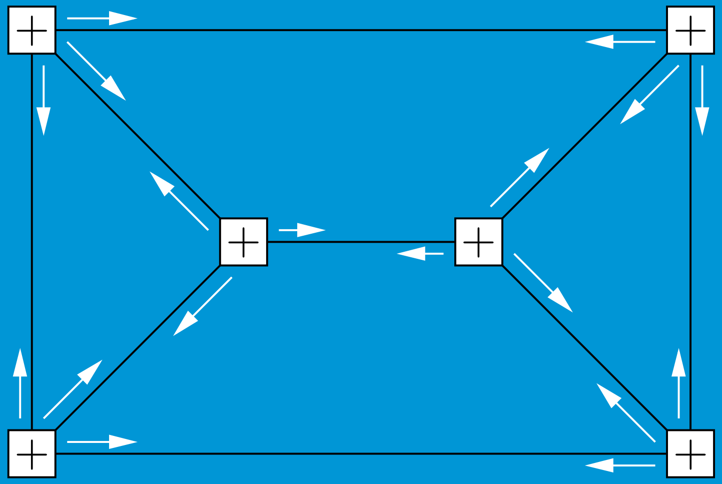


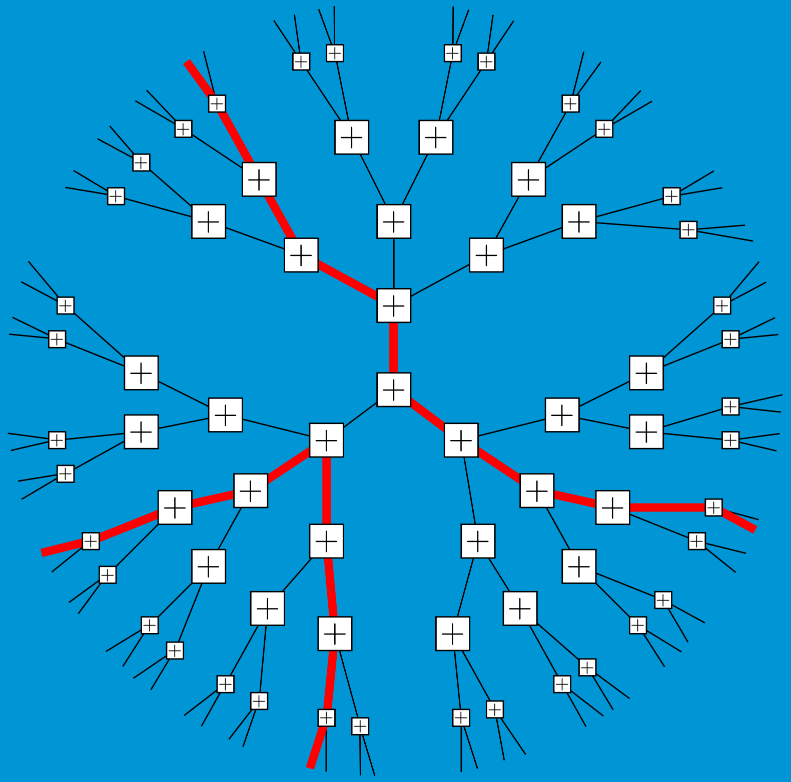
Graph Cover Hierarchy

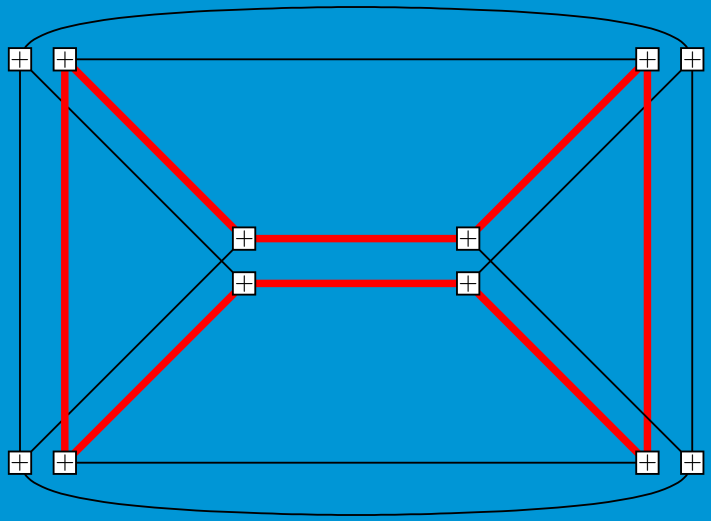


Graph Cover Hierarchy





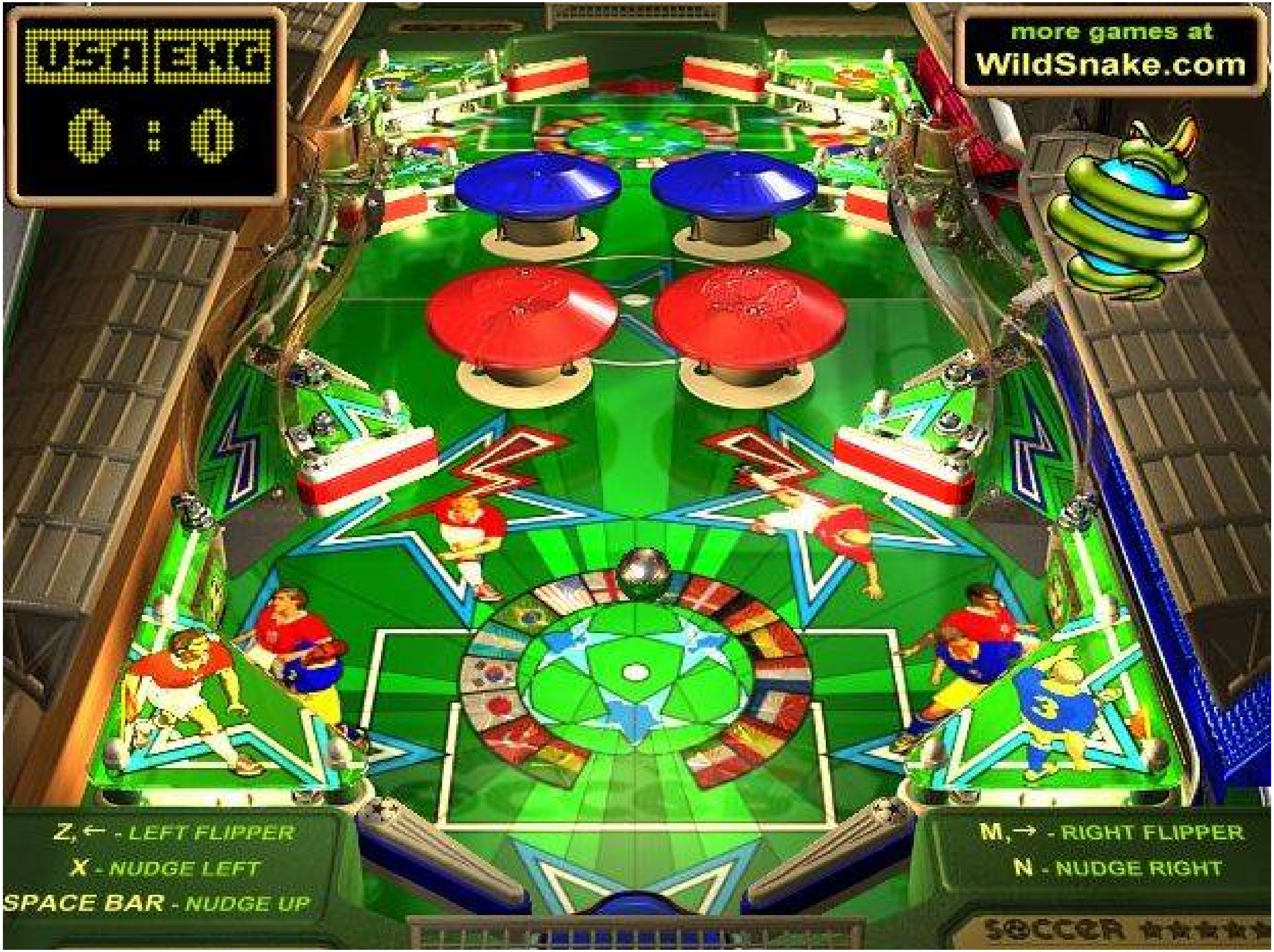




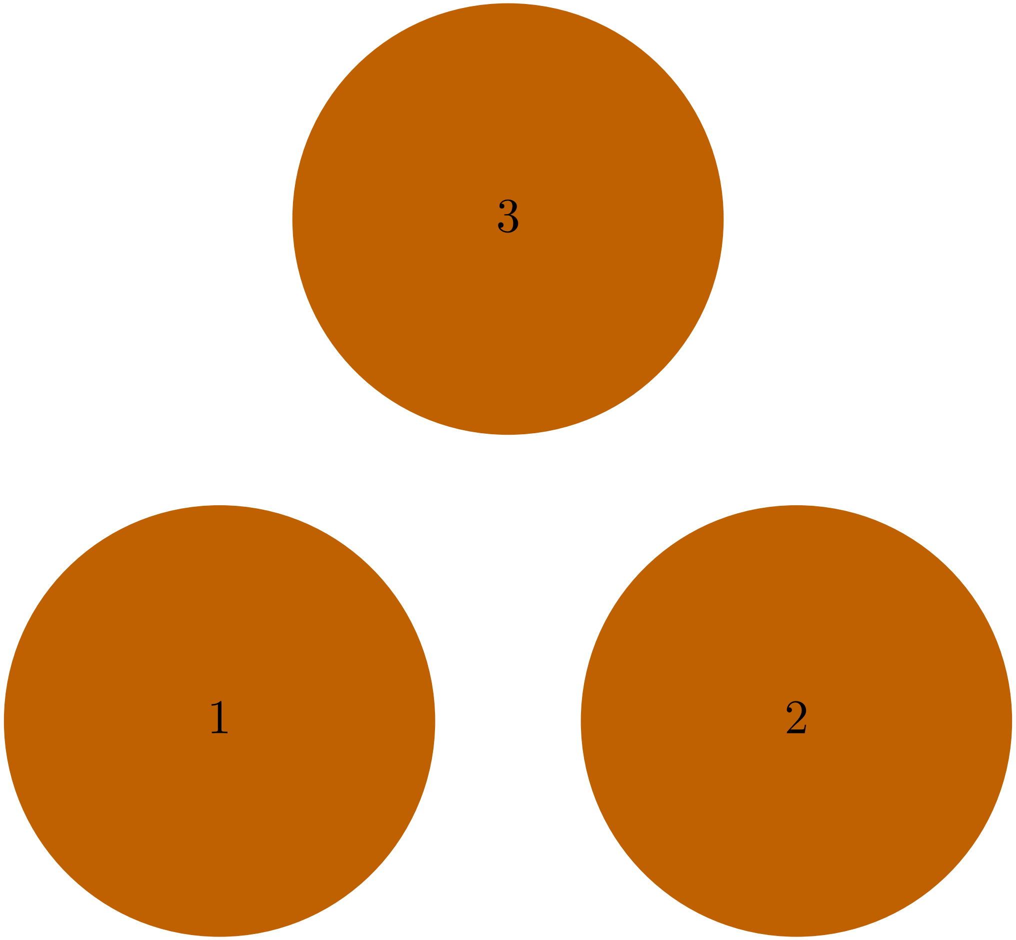
$$\zeta(\mathsf{V}_1,\ldots,\mathsf{V}_n) \;=\; \sum_{\mathbf{k}} \zeta_{\mathbf{k}} \mathbf{V}^{\mathbf{k}} \;=\; \prod_{[\Gamma]} \frac{1}{1-g(\Gamma,\mathbf{V})}$$

Pinball

Pinball

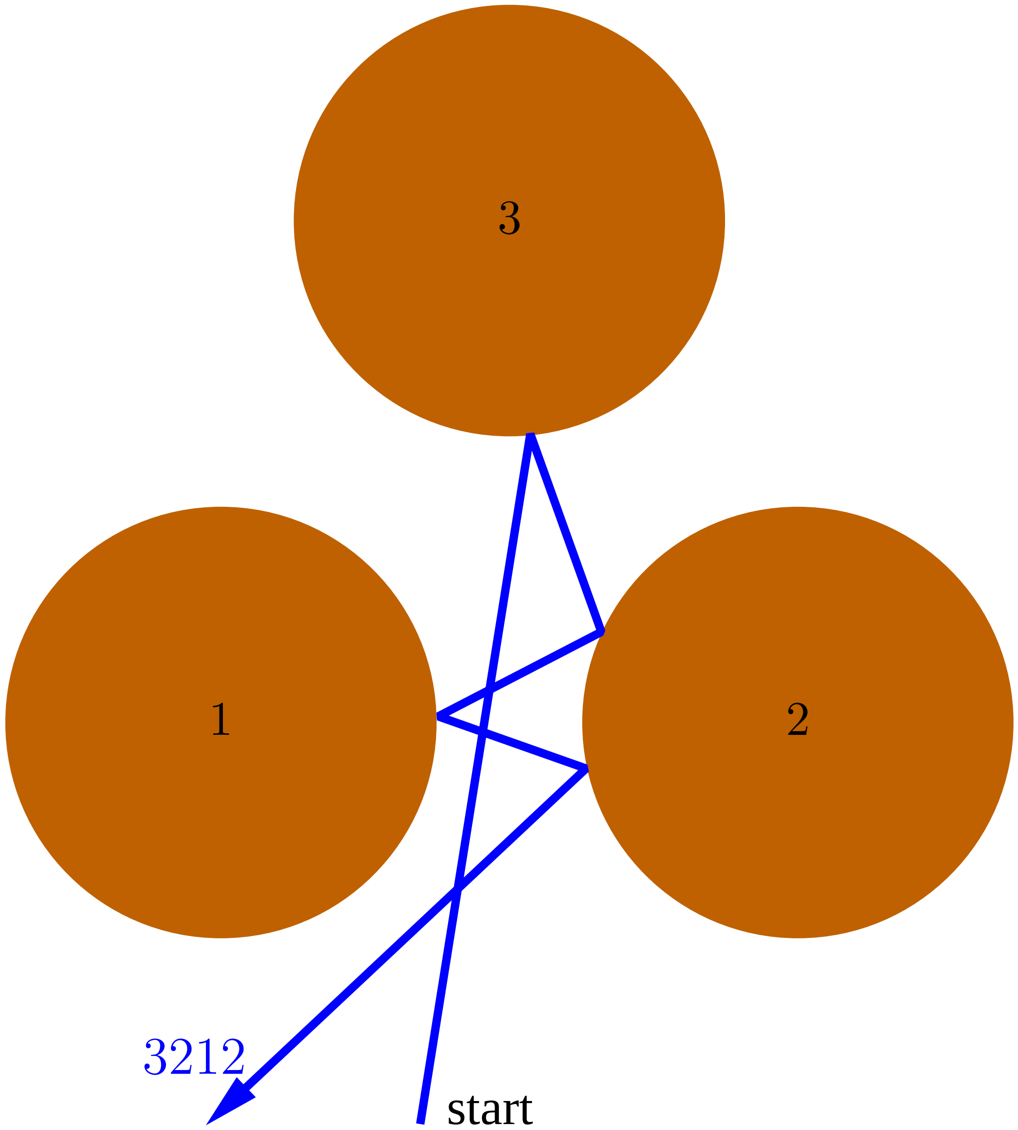

USA ENG
0 : 0
more games at
WildSnake.com
Z,← - LEFT FLIPPER
X - NUDGE LEFT
SPACE BAR - NUDGE UP
M,→ - RIGHT FLIPPER
N - NUDGE RIGHT
SOCCER

Pinball



[picture adapted from chaosbook.org]

Pinball



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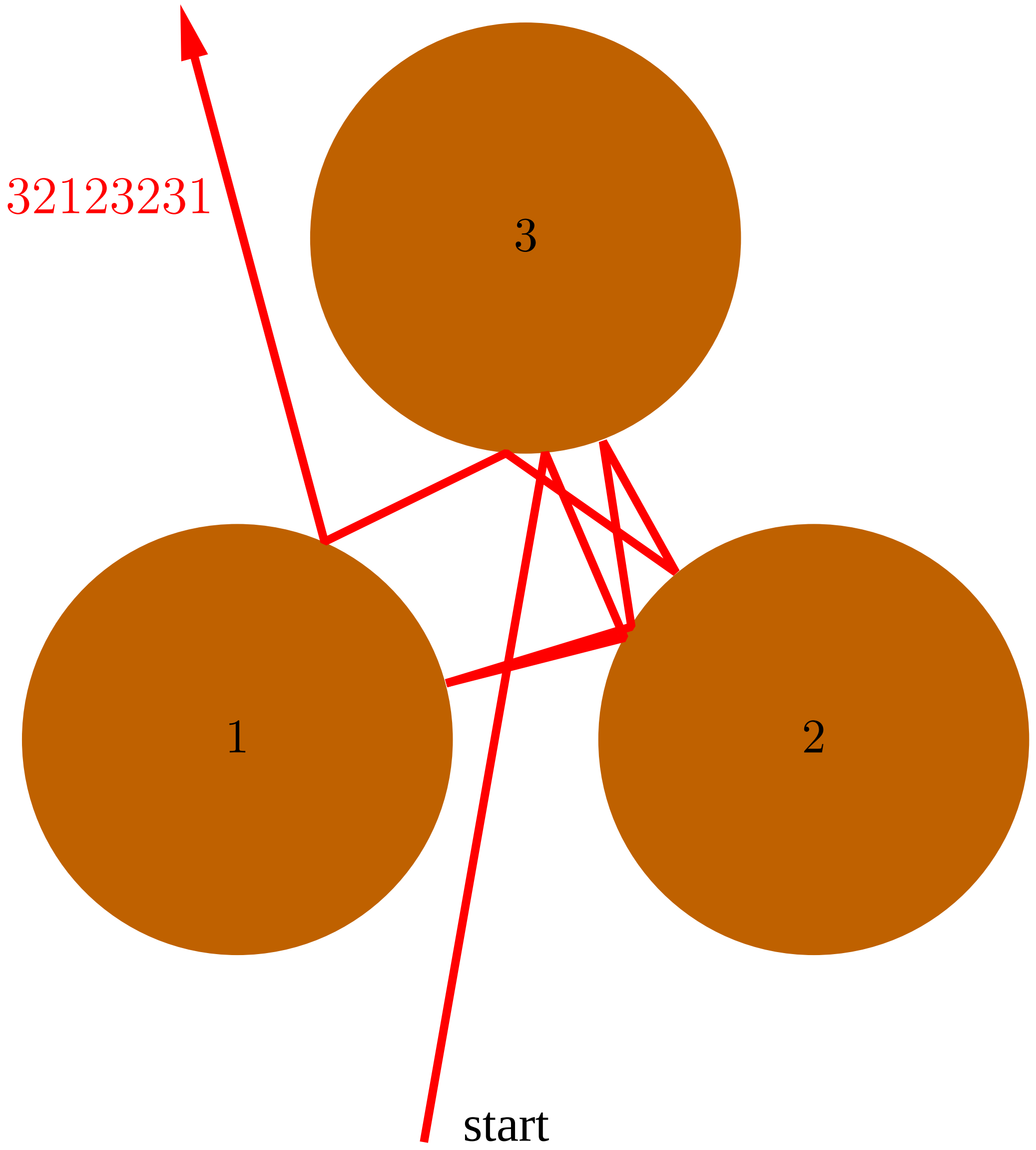

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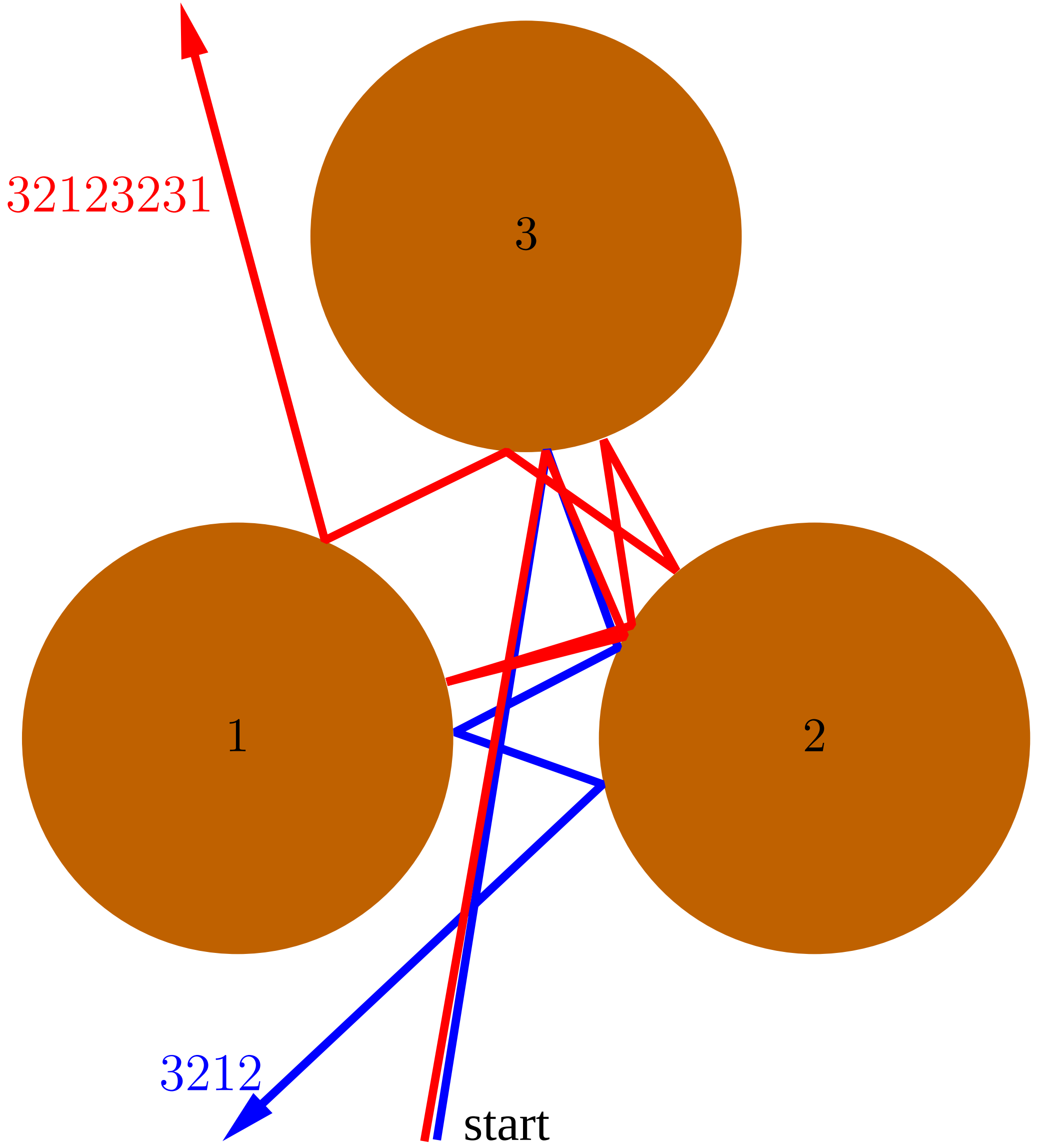

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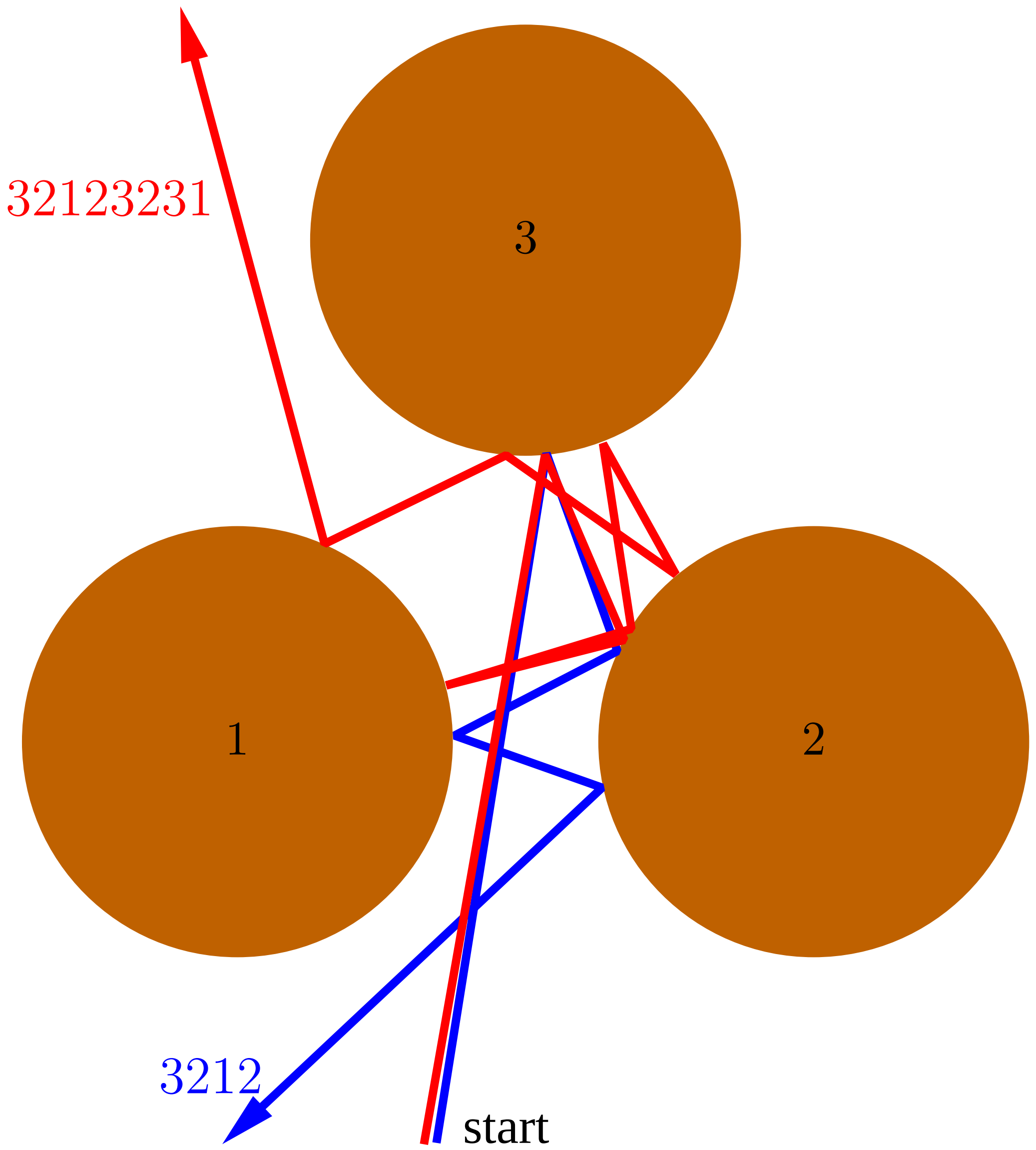
Pinball

The trajectories are difficult to predict.

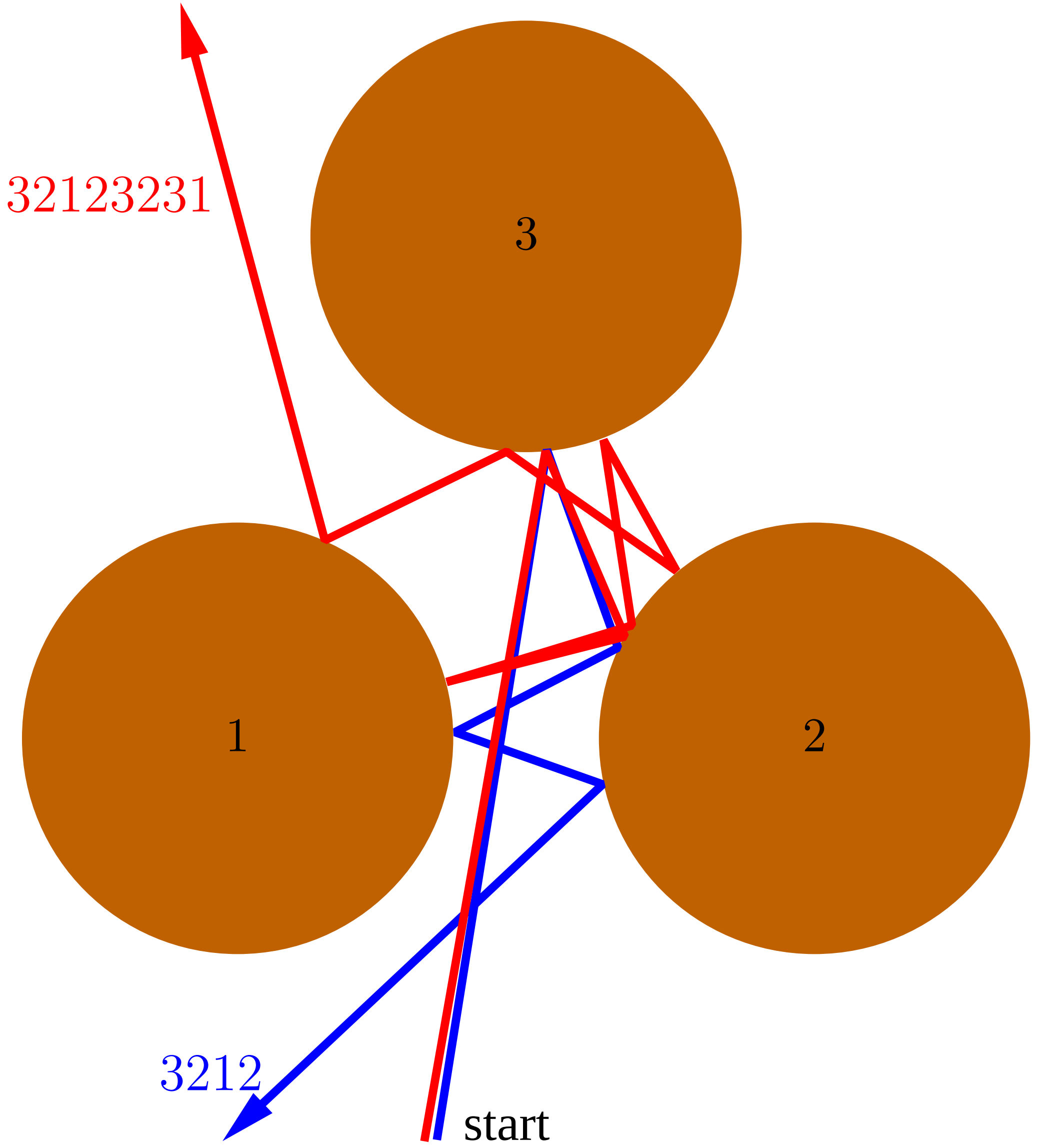


[picture adapted from chaosbook.org]

Pinball

The trajectories are difficult to predict.

“chaotic system”



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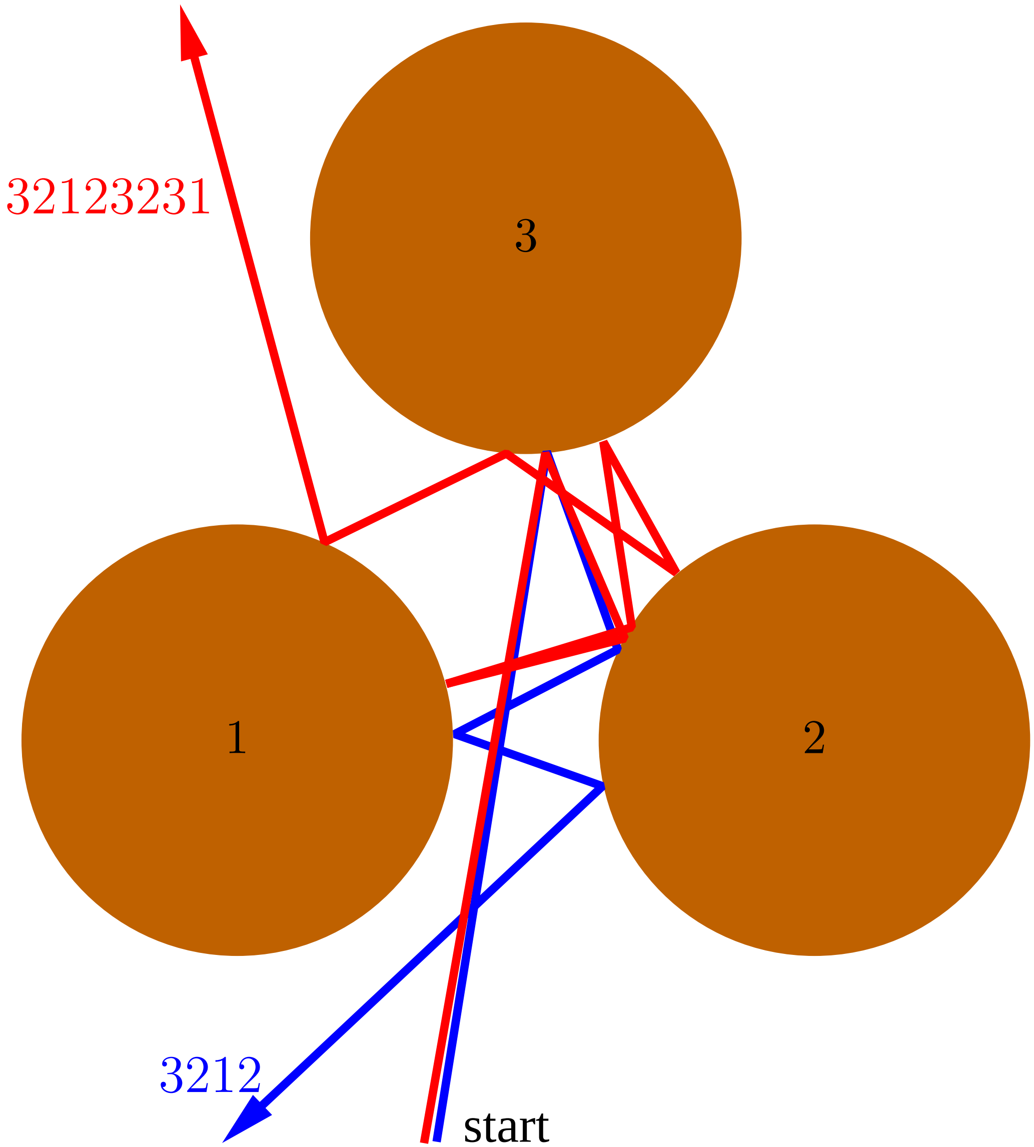
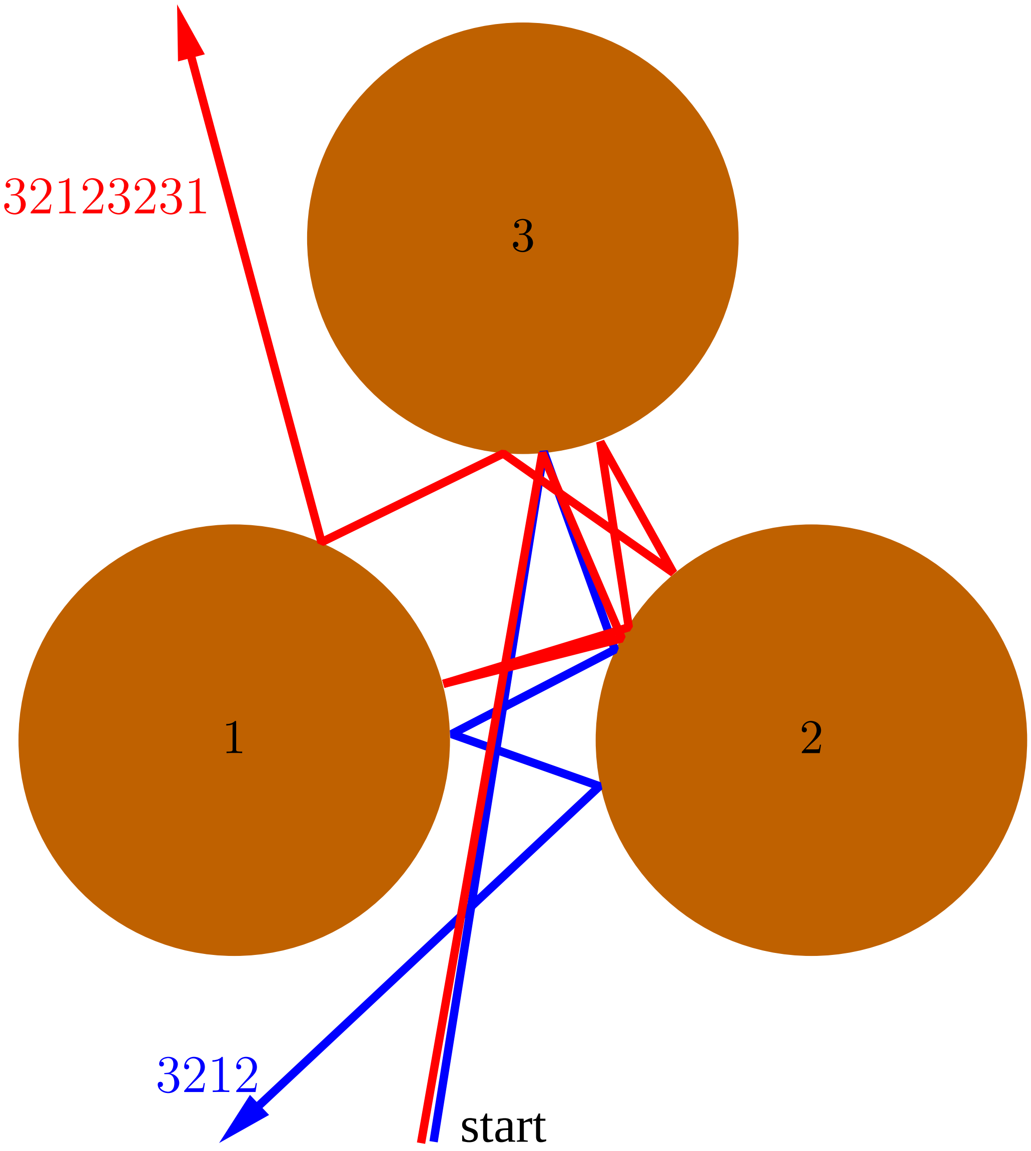


[picture adapted from chaosbook.org]

Pinball

The trajectory is difficult to predict.

However, quantities like the escape rate can be quantified.



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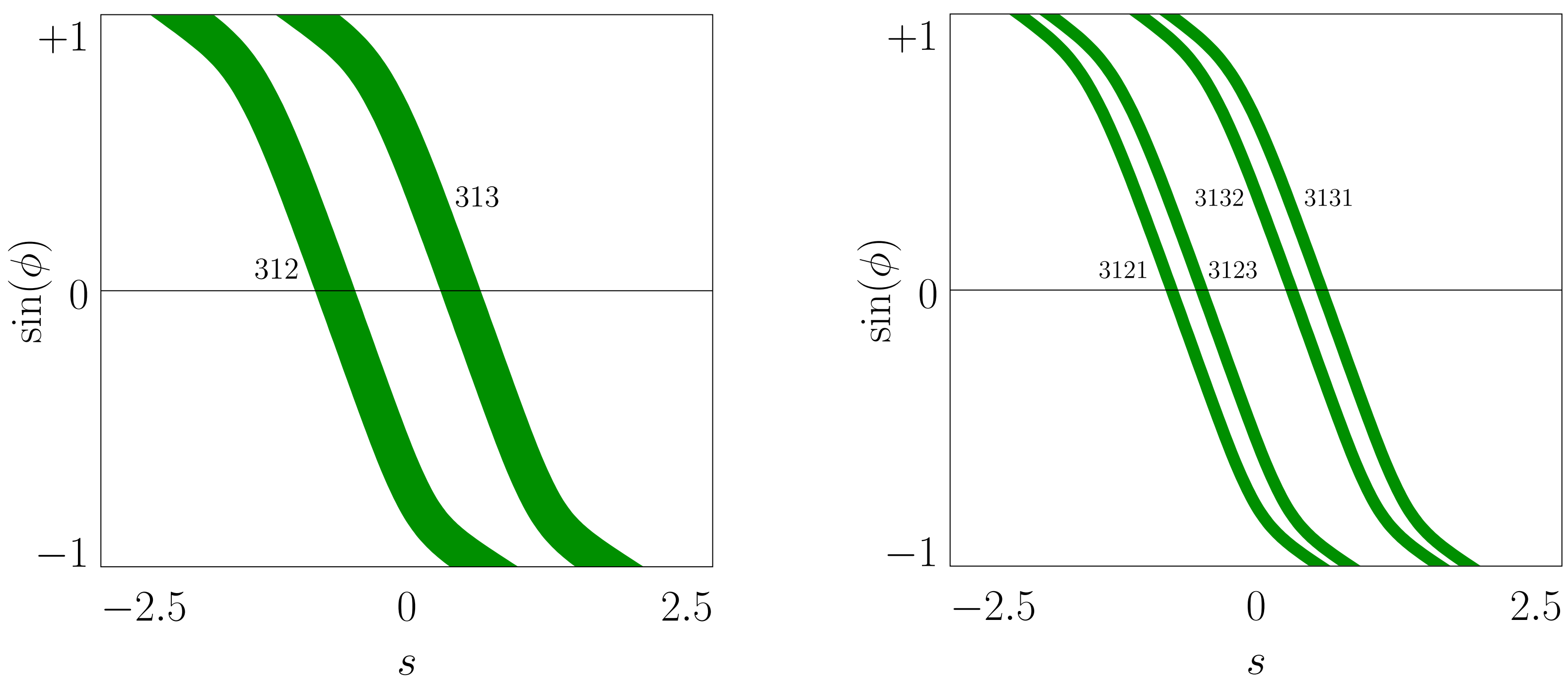
Pinball



$\mathcal{M}_3$: initial conditions for which the ball bounces at 3.

$\mathcal{M}_{31}$: initial conditions for which the ball bounces at 3, 1.

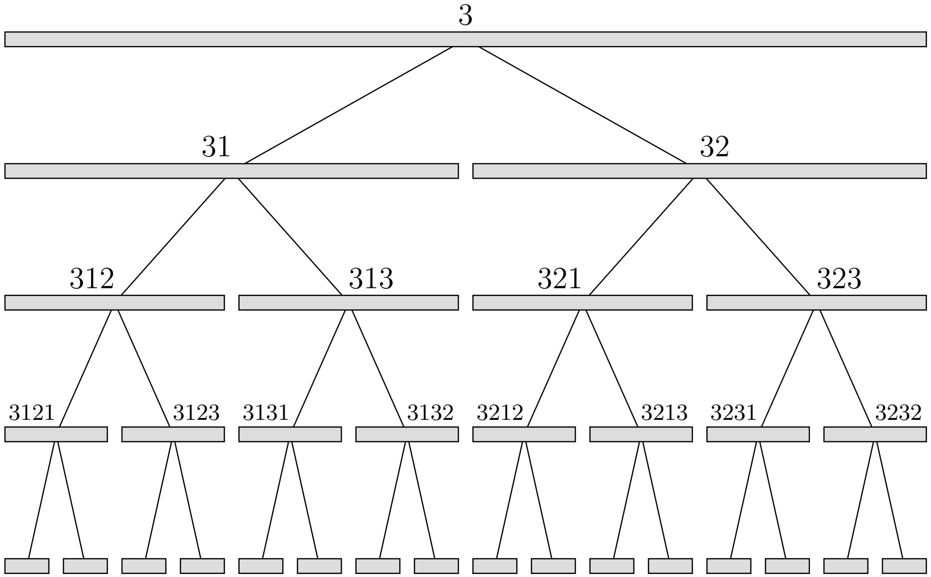
$\mathcal{M}_{312}$: initial conditions for which the ball bounces at 3, 1, 2.

$\mathcal{M}_{3121}$: initial conditions for which the ball bounces at 3, 1, 2, 1.

$\dots$: $\dots$

[picture adapted from chaosbook.org]

Pinball



[picture adapted from chaosbook.org]

Pinball

$$\hat{\theta}_1 = \frac{|\mathcal{M}_1|}{|\mathcal{M}|} + \frac{|\mathcal{M}_2|}{|\mathcal{M}|} + \frac{|\mathcal{M}_3|}{|\mathcal{M}|}$$

Pinball

$$\hat{\theta}_1 = \frac{|\mathcal{M}_1|}{|\mathcal{M}|} + \frac{|\mathcal{M}_2|}{|\mathcal{M}|} + \frac{|\mathcal{M}_3|}{|\mathcal{M}|}$$

$$\hat{\theta}_2 = \frac{|\mathcal{M}_{12}|}{|\mathcal{M}|} + \frac{|\mathcal{M}_{13}|}{|\mathcal{M}|} + \frac{|\mathcal{M}_{21}|}{|\mathcal{M}|} + \frac{|\mathcal{M}_{22}|}{|\mathcal{M}|} + \frac{|\mathcal{M}_{31}|}{|\mathcal{M}|} + \frac{|\mathcal{M}_{32}|}{|\mathcal{M}|}$$

Pinball

$$
\begin{aligned}
\hat{\theta}_1 &= \frac{|\mathcal{M}_1|}{|\mathcal{M}|} + \frac{|\mathcal{M}_2|}{|\mathcal{M}|} + \frac{|\mathcal{M}_3|}{|\mathcal{M}|} \\
\hat{\theta}_2 &= \frac{|\mathcal{M}_{12}|}{|\mathcal{M}|} + \frac{|\mathcal{M}_{13}|}{|\mathcal{M}|} + \frac{|\mathcal{M}_{21}|}{|\mathcal{M}|} + \frac{|\mathcal{M}_{22}|}{|\mathcal{M}|} + \frac{|\mathcal{M}_{31}|}{|\mathcal{M}|} + \frac{|\mathcal{M}_{32}|}{|\mathcal{M}|} \\
\vdots & \qquad \vdots
\end{aligned}
$$

Pinball

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$$\vdots \qquad \vdots$$

$$\hat{\theta}_n = \sum_{\text{sequence } \mathbf{s} \text{ of length } n} \frac{|\mathcal{M}_{\mathbf{s}}|}{|\mathcal{M}|}$$

Pinball

$$
\begin{aligned}
\hat{\theta}_1 &= \frac{|\mathcal{M}_1|}{|\mathcal{M}|} + \frac{|\mathcal{M}_2|}{|\mathcal{M}|} + \frac{|\mathcal{M}_3|}{|\mathcal{M}|} \\
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$$

Pinball

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$$\vdots \qquad \vdots$$

$$\hat{\theta}_n = \sum_{\text{sequence } \mathbf{s} \text{ of length } n} \frac{|\mathcal{M}_{\mathbf{s}}|}{|\mathcal{M}|}$$

$$\vdots \qquad \vdots$$

$$\frac{\hat{\theta}_{n+1}}{\hat{\theta}_n} = \exp(-\gamma_n)$$

Pinball

$$\hat{\theta}_1 = \frac{|\mathcal{M}_1|}{|\mathcal{M}|} + \frac{|\mathcal{M}_2|}{|\mathcal{M}|} + \frac{|\mathcal{M}_3|}{|\mathcal{M}|}$$

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$$\vdots \qquad \vdots$$

$$\hat{\theta}_n = \sum_{\text{sequence } \mathbf{s} \text{ of length } n} \frac{|\mathcal{M}_{\mathbf{s}}|}{|\mathcal{M}|}$$

$$\vdots \qquad \vdots$$

$$\frac{\hat{\theta}_{n+1}}{\hat{\theta}_n} = \exp(-\gamma_n) \quad \rightarrow \quad \exp(-\gamma)$$

Pinball

$$\hat{\theta}_1 = \frac{|\mathcal{M}_1|}{|\mathcal{M}|} + \frac{|\mathcal{M}_2|}{|\mathcal{M}|} + \frac{|\mathcal{M}_3|}{|\mathcal{M}|}$$

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$$\vdots \qquad \vdots$$

$$\hat{\theta}_n = \sum_{\text{sequence } \mathrm{s} \text{ of length } n} \frac{|\mathcal{M}_{\mathbf{s}}|}{|\mathcal{M}|}$$

$$\vdots \qquad \vdots$$

$$\frac{\hat{\theta}_{n+1}}{\hat{\theta}_n} = \exp(-\gamma_n) \quad \to \quad \exp(-\gamma)$$

γ : escape rate

Pinball

- Ideally, we compute $\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3, \ldots$, and determine from this γ.

Pinball

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Pinball

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- Note that the **power series**

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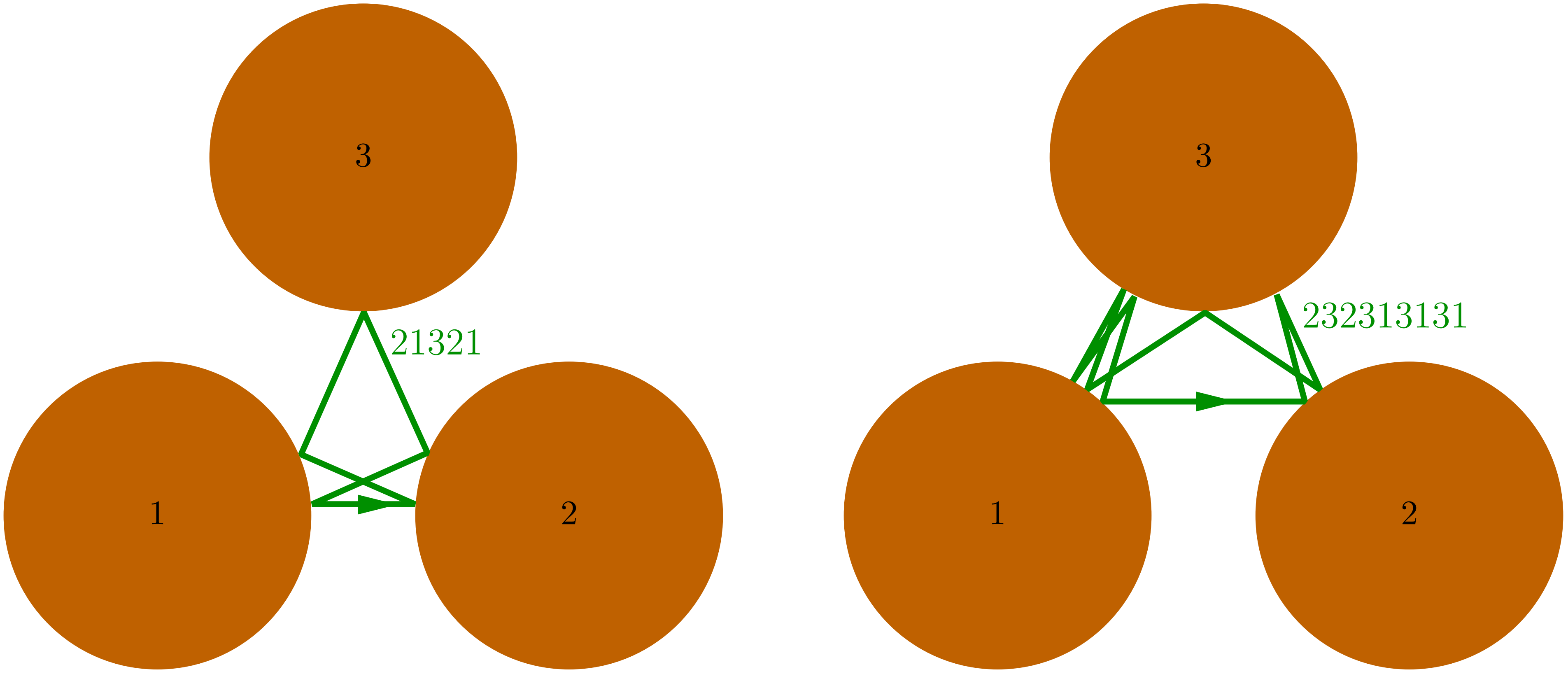
- Alternative approach: set up a **new power series**

$$\theta(z) = \sum_{n=1}^{\infty} \theta_n z^n$$

so that its **convergence radius** equals $\exp(\gamma)$.

Pinball

Main idea: look at periodic trajectories.



[picture adapted from chaosbook.org]

Pinball

Pinball

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Pinball

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- Let the periodic trajectory $\mathbf{s}$ go through $x_{\mathbf{s}}$.
- Let the periodic trajectory $\mathbf{s}$ have period $T_{\mathrm{p},\mathbf{s}}$.
- Then define

$$
\begin{aligned}
\theta(z) &\triangleq \sum_{n=1}^{\infty} z^n \sum_{\text{sequence } \mathbf{s} \text{ of length } n} \frac{1}{|\Lambda_{\mathbf{s}}|} \\
&= \frac{z^1}{|\Lambda_1|} + \frac{z^1}{|\Lambda_2|} + \frac{z^1}{|\Lambda_3|} + \frac{z^2}{|\Lambda_{12}|} + \frac{z^2}{|\Lambda_{13}|} + \cdots
\end{aligned}
$$

where Λ_{s} is the unstable eigenvalue of the Jacobian matrix $J^t(x_i)$ evaluated for $t = T_{\mathrm{p},\mathbf{s}}$. (Due to the low dimensionality, the Jacobian can have at most one unstable eigenvalue for the present setup.)

Pinball

- $\theta(z)$ can be rewritten as follows:

$$\theta(z) = \sum_{\text{prime cycle } \mathbf{p}} n_{\mathbf{p}} \sum_{r=1}^{\infty} \left(\frac{z^{n_{\mathbf{p}}}}{|\Lambda_{\mathbf{p}}|} \right)^{r}$$

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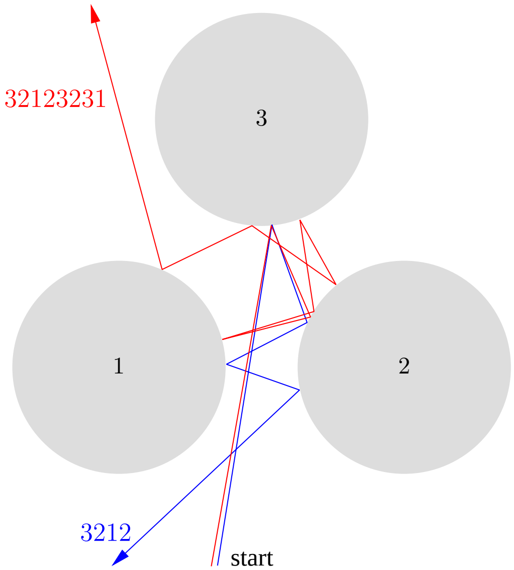
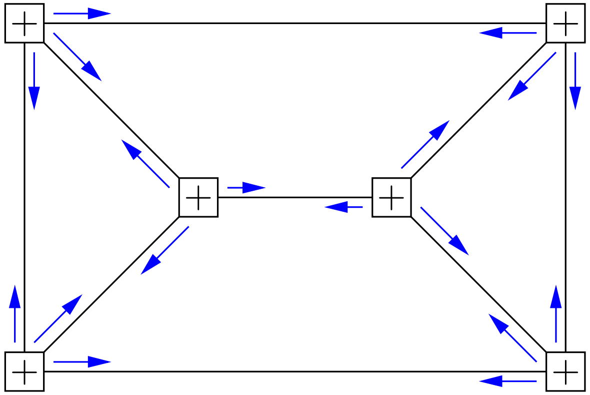
- Note:

$$\theta(z) = z \frac{\mathrm{d}}{\mathrm{d}z} \log\left(\zeta(z)\right)$$

Analogy Pinball vs. MPI Decoding

 pinball	 message-passing iterative decoding of cycle codes
trajectory	minimial deviation in computation trees
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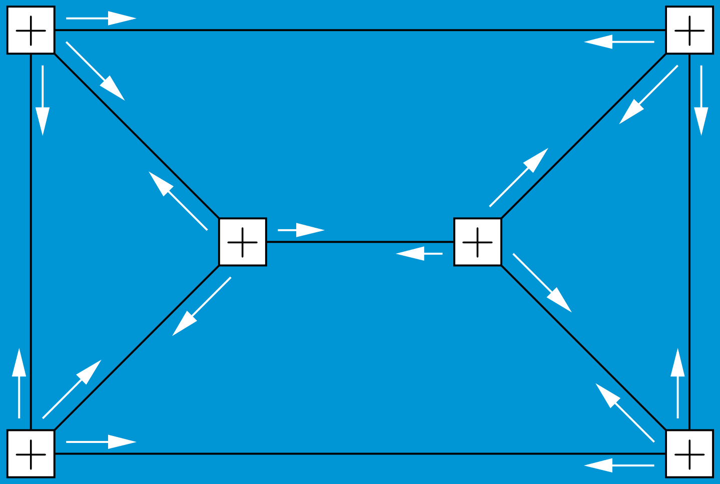
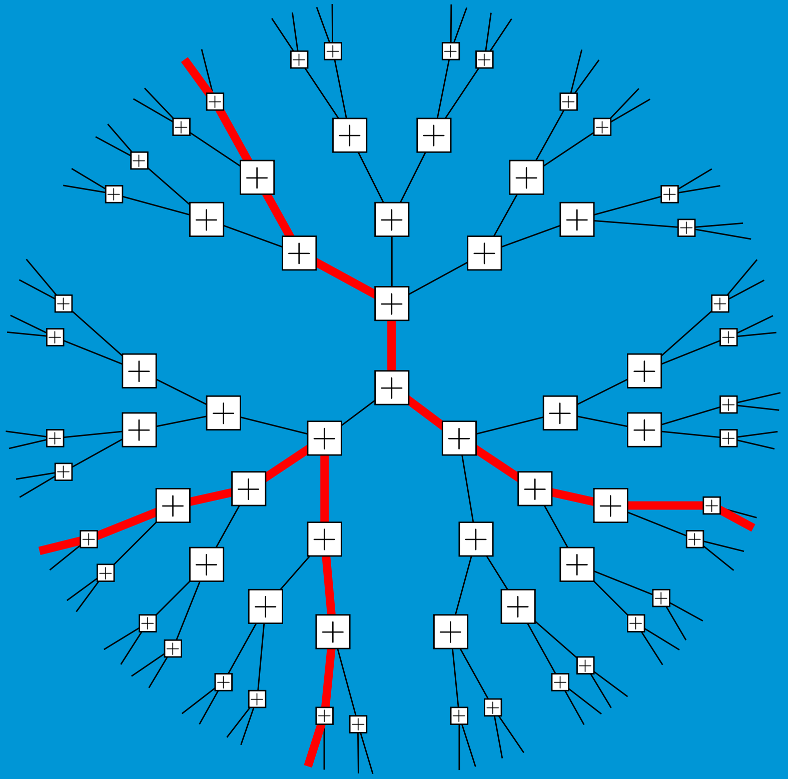
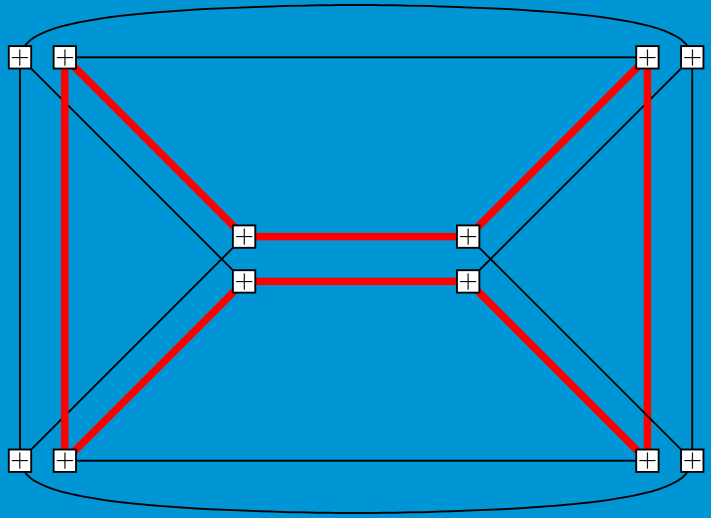


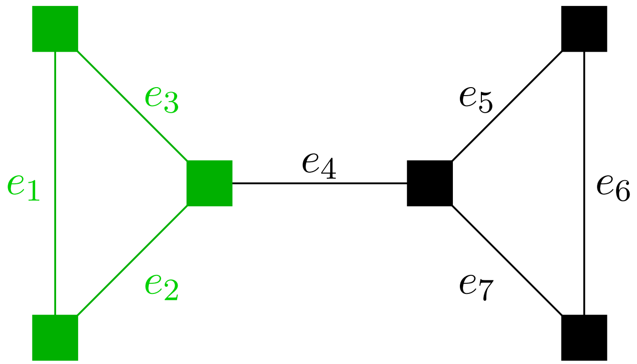




$$\zeta(\mathsf{V}_1,\ldots,\mathsf{V}_n) = \sum_{\mathbf{k}} \zeta_{\mathbf{k}} \mathbf{V}^{\mathbf{k}} = \prod_{[\Gamma]} \frac{1}{1-g(\Gamma,\mathbf{V})}$$

The Edge Zeta Function of a Graph

Definition (Hashimoto, see also Stark/Terras):



Here: $\Gamma = (e_1, e_2, e_3)$

Let Γ be a path in a graph X with edge-set E; write

$$\Gamma = (e_{i_1}, \ldots, e_{i_k})$$

to indicate that Γ begins with the edge e_{i_1} and ends with the edge e_{i_k}.

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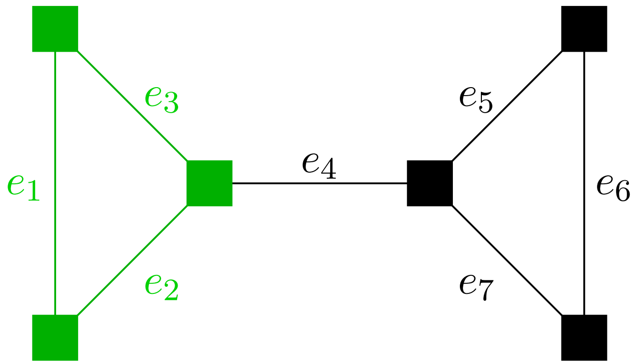

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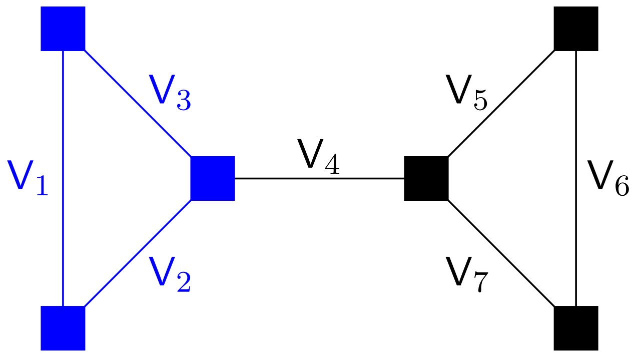

Here: $g(\Gamma, \mathbf{V}) = \mathsf{V}_1\mathsf{V}_2\mathsf{V}_3$

The monomial of Γ is given by

$$g(\Gamma, \mathbf{V}) \triangleq \mathsf{V}_{i_1} \cdots \mathsf{V}_{i_k},$$

where the V_i's are indeterminates.

The Edge Zeta Function of a Graph

Definition (Hashimoto, see also Stark/Terras):

The edge zeta function of X is defined to be the power series

$$\zeta_X(\mathbf{V}) = \zeta_X(\mathsf{V}_1, \ldots, \mathsf{V}_n) \in \mathbb{Z}[[\mathsf{V}_1, \ldots, \mathsf{V}_n]]$$

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$$\zeta_X(\mathbf{V}) = \zeta_X(\mathsf{V}_1, \ldots, \mathsf{V}_n) = \prod_{[\Gamma] \in A(X)} \frac{1}{1 - g(\Gamma, \mathbf{V})},$$

where $A(X)$ is the collection of equivalence classes of

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backtrackless, tailless, primitive cycles in X.

Note: unless X contains only one cycle,

the set $A(X)$ will be countably infinite.

The Edge Zeta Function of a Graph

Theorem (Bass):

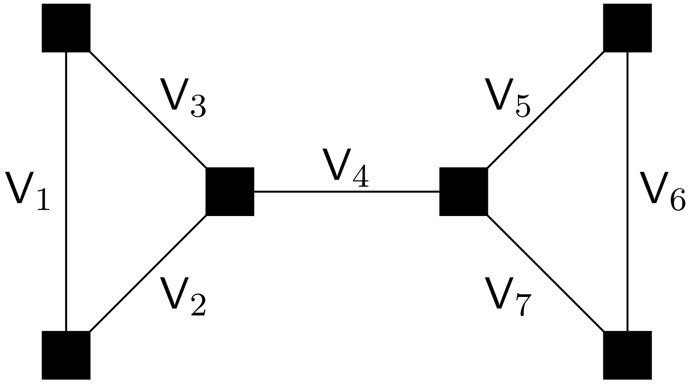
- The edge zeta function $\zeta_X(\mathsf{V}_1, \ldots, \mathsf{V}_n)$ is a rational function.
- More precisely, for any directed graph $\vec{X}$ of X, we have

$$\zeta_X(\mathsf{V}_1, \ldots, \mathsf{V}_n) = \frac{1}{\det\left(\mathbf{I} - \mathbf{V}\mathbf{M}(\vec{X})\right)} = \frac{1}{\det\left(\mathbf{I} - \mathbf{M}(\vec{X})\mathbf{V}\right)}$$

  where

  - $\mathbf{I}$ is the identity matrix of size $2n$,
  - $\mathbf{V} = \text{diag}(\mathsf{V}_1, \ldots, \mathsf{V}_n, \mathsf{V}_1, \ldots, \mathsf{V}_n)$ is a diagonal matrix of indeterminants.
  - $\mathbf{M}(\vec{X})$ is a $2n \times 2n$ matrix derived from some directed graph version $\vec{X}$ of X.

Example of Edge Zeta Function

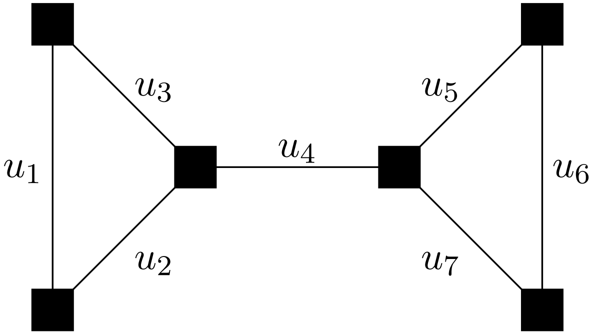


This normal graph N has the following edge zeta function:

$$\zeta_N(V_1, \ldots, V_7) = \frac{1}{\det(\mathbf{I}_{14} - \mathbf{VM})}$$

$$= \frac{1}{\begin{array}{l} 1 - 2V_1V_2V_3 + V_1^2V_2^2V_3^2 - 2V_5V_6V_7 + 4V_1V_2V_3V_5V_6V_7 - 2V_1^2V_2^2V_3^2V_5V_6V_7 \\ -4V_1V_2V_3V_4^2V_5V_6V_7 + 4V_1^2V_2^2V_3^2V_4^2V_5V_6V_7 + V_5^2V_6^2V_7^2 - 2V_1V_2V_3V_5^2V_6^2V_7^2 \\ +V_1^2V_2^2V_3^2V_5^2V_6^2V_7^2 + 4V_1V_2V_3V_4^2V_5^2V_6^2V_7^2 - 4V_1^2V_2^2V_3^2V_4^2V_5^2V_6^2V_7^2 \end{array}}$$

Example of Edge Zeta Function

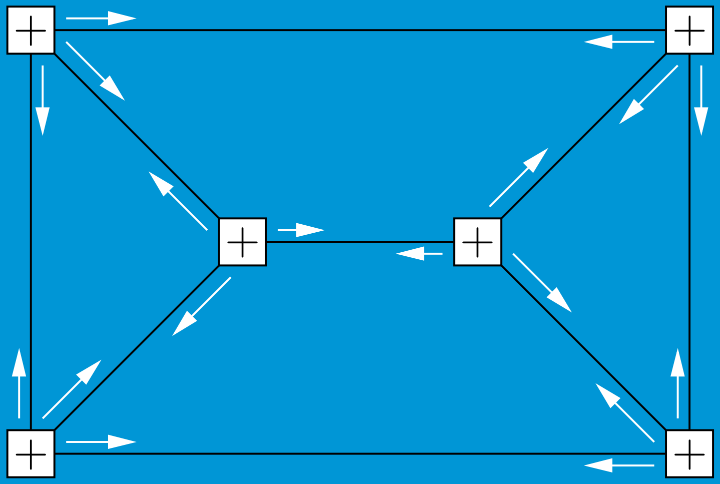


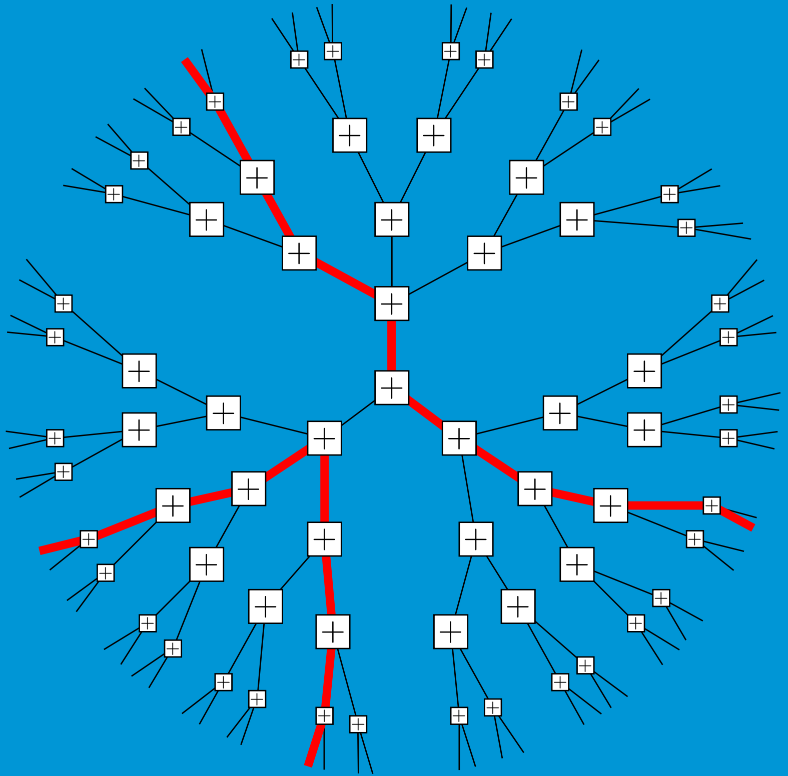
The Taylor series exansion is

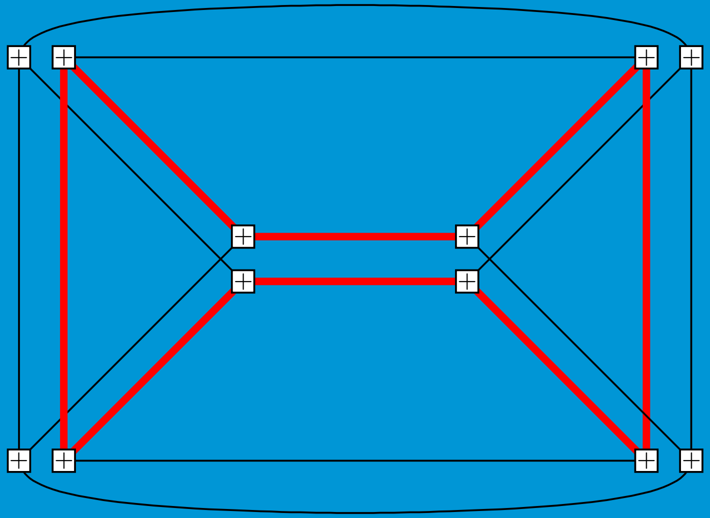
$$\begin{aligned}\zeta_N(V_1, \ldots, V_7) &= 1 + 2V_1V_2V_3 + 3V_1^2V_2^2V_3^2 + 2V_5V_6V_7 \\ &\quad + 4V_1V_2V_3V_5V_6V_7 + 6V_1^2V_2^2V_3^2V_5V_6V_7 \\ &\quad + 4V_1V_2V_3V_4^2V_5V_6V_7 + 12V_1^2V_2^2V_3^2V_4^2V_5V_6V_7 \\ &\quad + \cdots\end{aligned}$$

We get the following exponent vectors:

$(0, 0, 0, 0, 0, 0, 0)$	codeword
$(1, 1, 1, 0, 0, 0, 0)$	codeword
$(2, 2, 2, 0, 0, 0, 0)$	pseudo-codeword (in $\mathbb{Z}$-span)
$(0, 0, 0, 0, 1, 1, 1)$	codeword
$(1, 1, 1, 0, 1, 1, 1)$	codeword
$(2, 2, 2, 0, 1, 1, 1)$	pseudo-codeword (in $\mathbb{Z}$-span)
$(1, 1, 1, 2, 1, 1, 1)$	pseudo-codeword (not in $\mathbb{Z}$-span)
$(2, 2, 2, 2, 1, 1, 1)$	pseudo-codeword (in $\mathbb{Z}$-span)







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New Theorem for Cycle Codes

Rough statement:

$$\begin{pmatrix} \text{region of convergence} \\ \text{of sum-product algorithm} \\ \text{to the all-zero codeword} \end{pmatrix} = \begin{pmatrix} \text{region of convergence} \\ \text{of the edge zeta function} \end{pmatrix}$$

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$$\begin{pmatrix}\text{The sum-product algorithm}\\ \text{converges to the all-zero codeword}\\ \text{for the log-likelihood vector } \lambda\end{pmatrix} \Leftrightarrow \begin{pmatrix}\text{V is in the region of convergence}\\ \text{of the edge zeta function,}\\ \text{where } \mathrm{V}_e = \exp(-\lambda_e) \ \forall e\end{pmatrix}$$

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Note: global convergence result!

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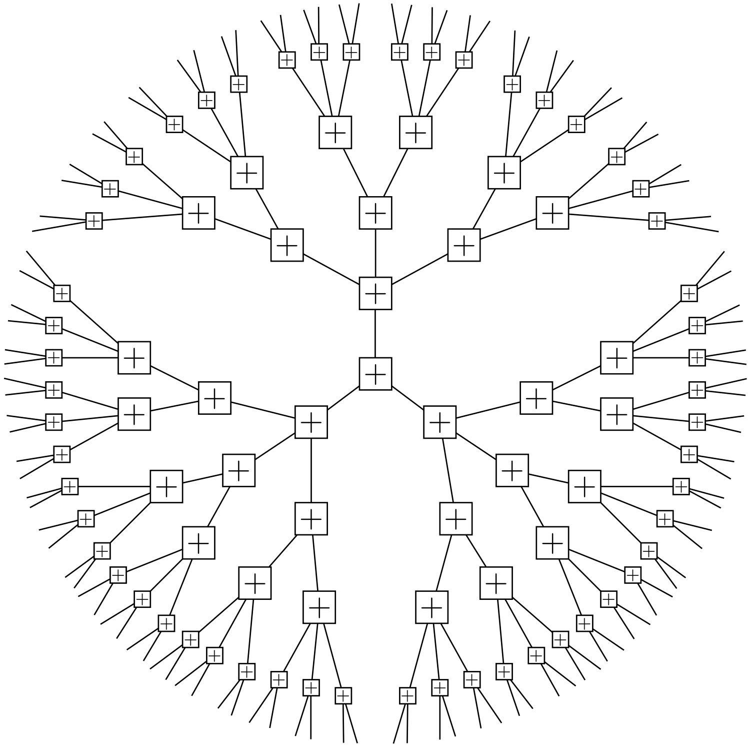
Corollary:

The region of all log-likelihood vectors λ
for which the sum-product algorithm converges
to the all-zero codeword
is given by a determinantal expression.

Some intuition behind this statement



$$\zeta_N(\mathsf{V}_1,\ldots,\mathsf{V}_n) \triangleq \prod_{[\Gamma]\in A(N)} \frac{1}{1-g(\Gamma,\mathbf{V})}$$

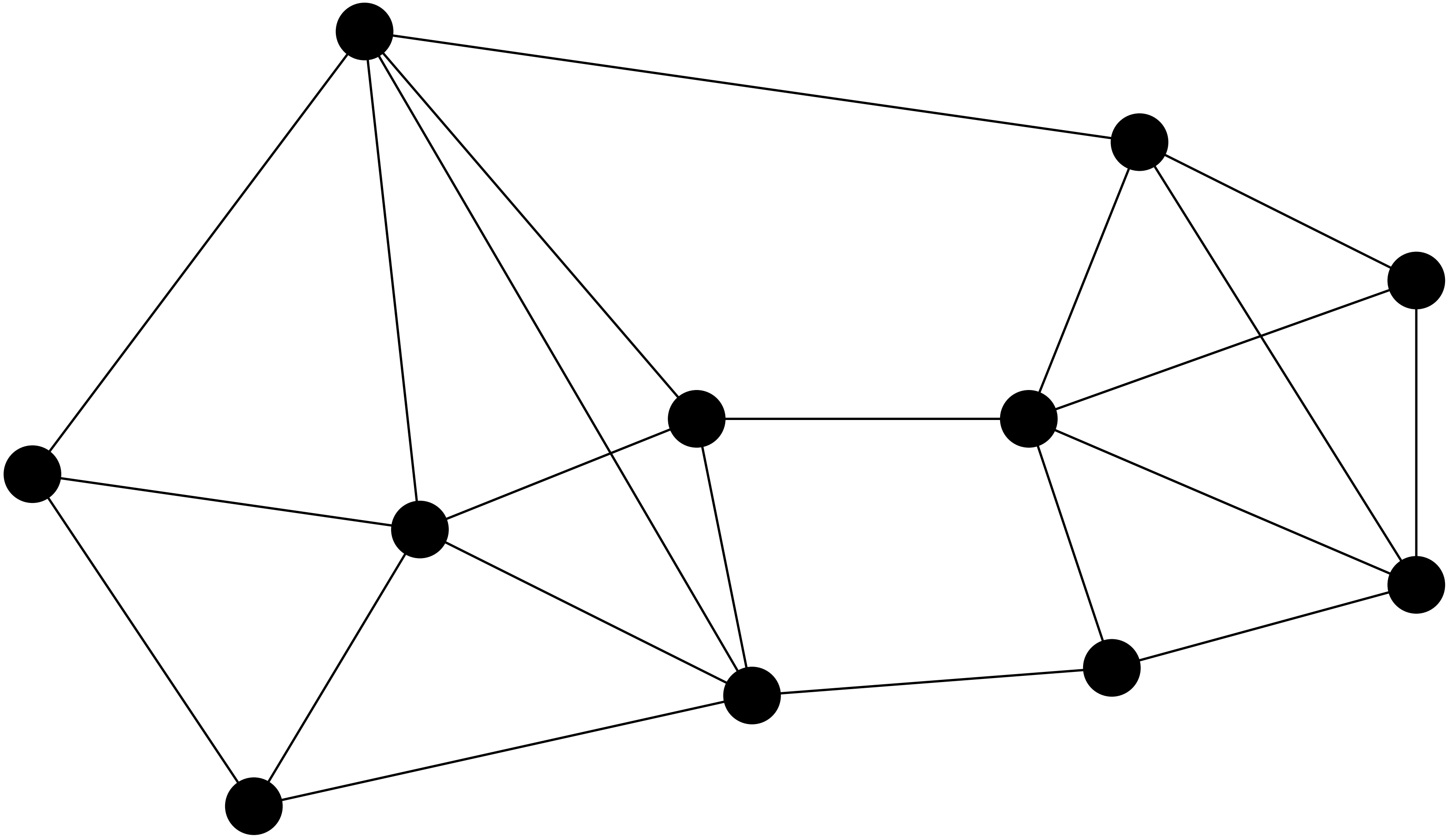
$$= \frac{1}{\det\left(\mathbf{I}-\mathbf{M}(\vec{N})\mathbf{V}\right)}$$

It turns out that key objects for analyzing computation trees of the normal factor graph N are

$$(\mathbf{M}(\vec{N})\mathbf{V})^k, \quad k \geq 0.$$

Community detection in networks

Community Detection in Networks



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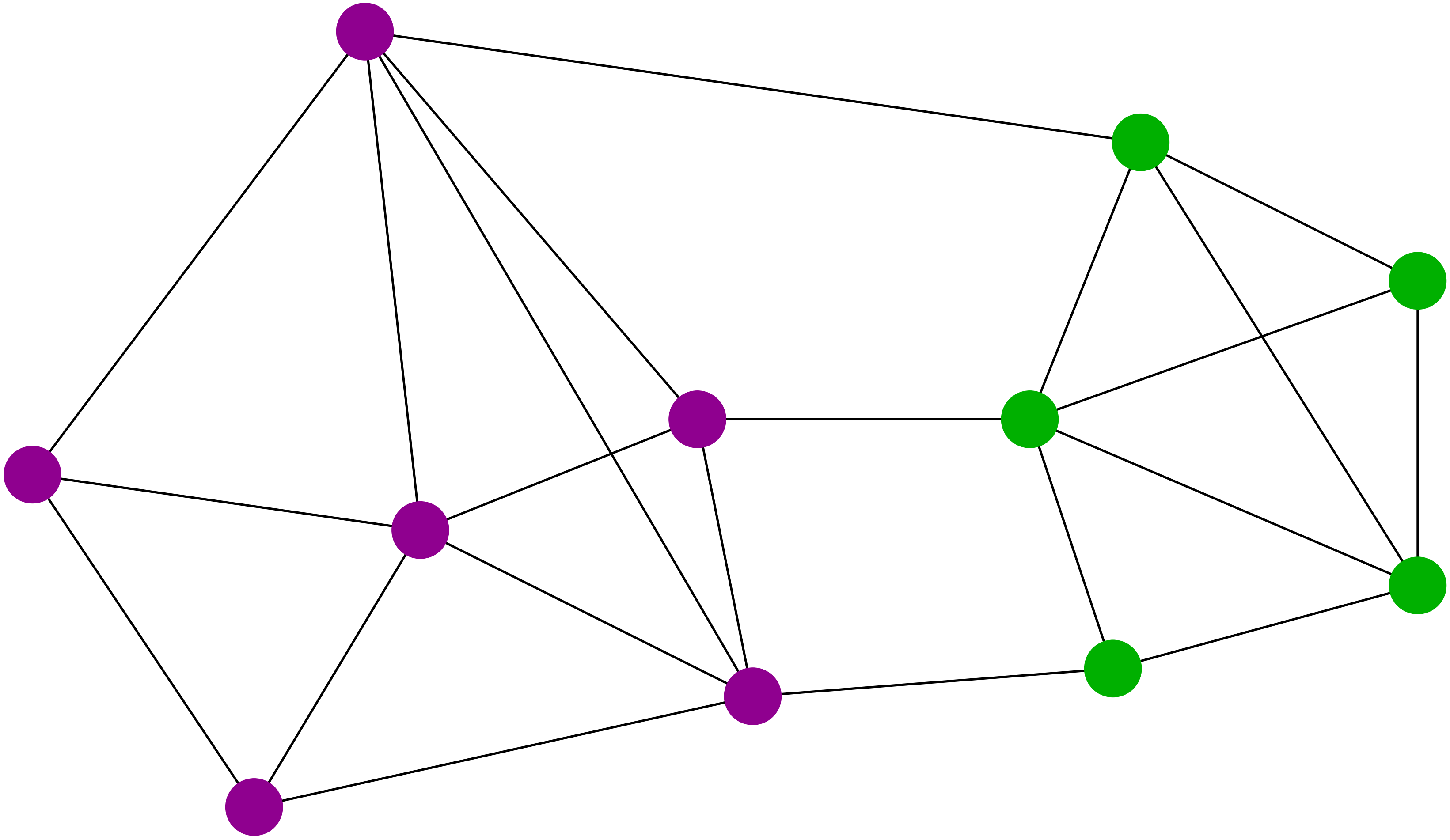

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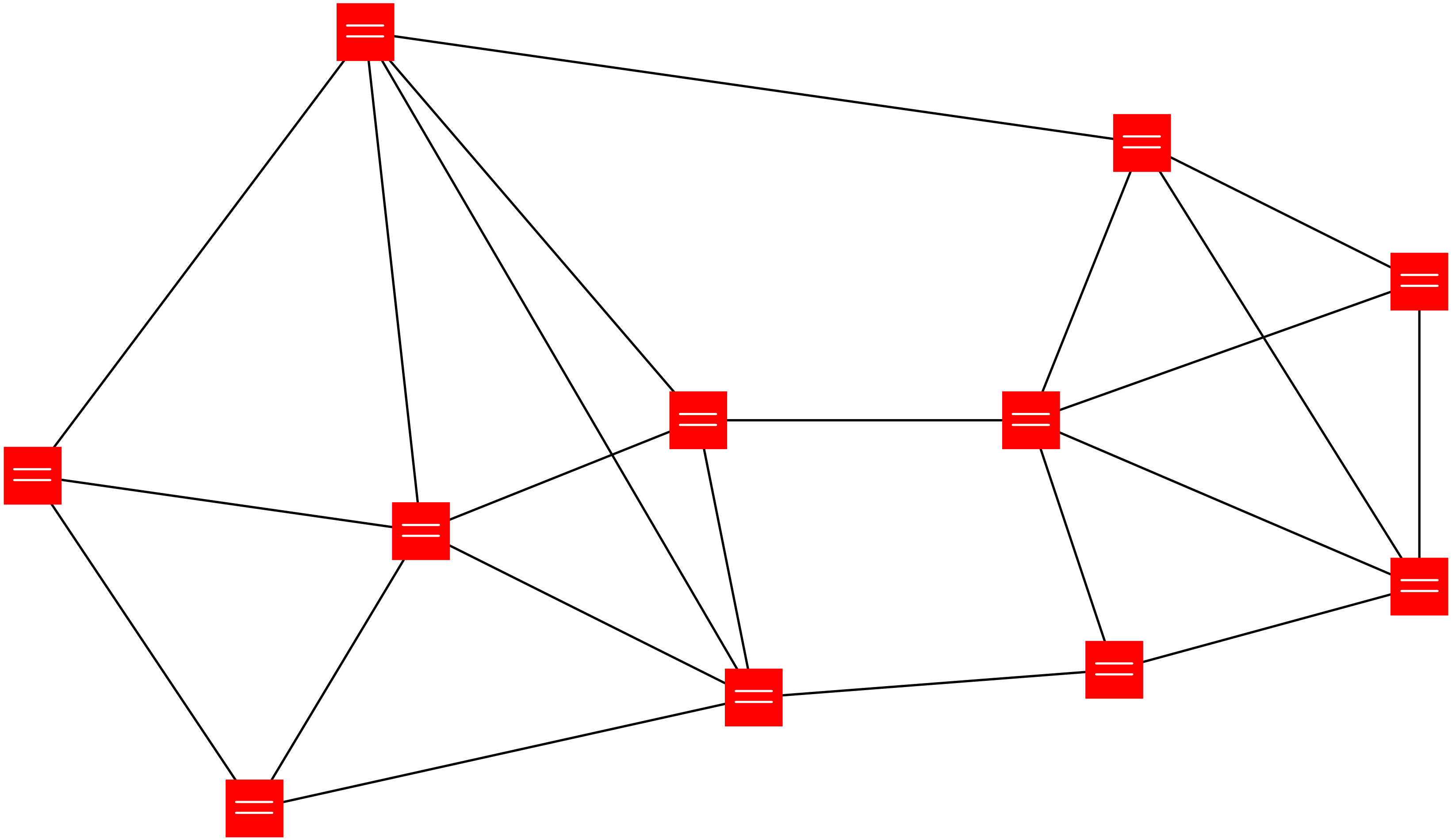

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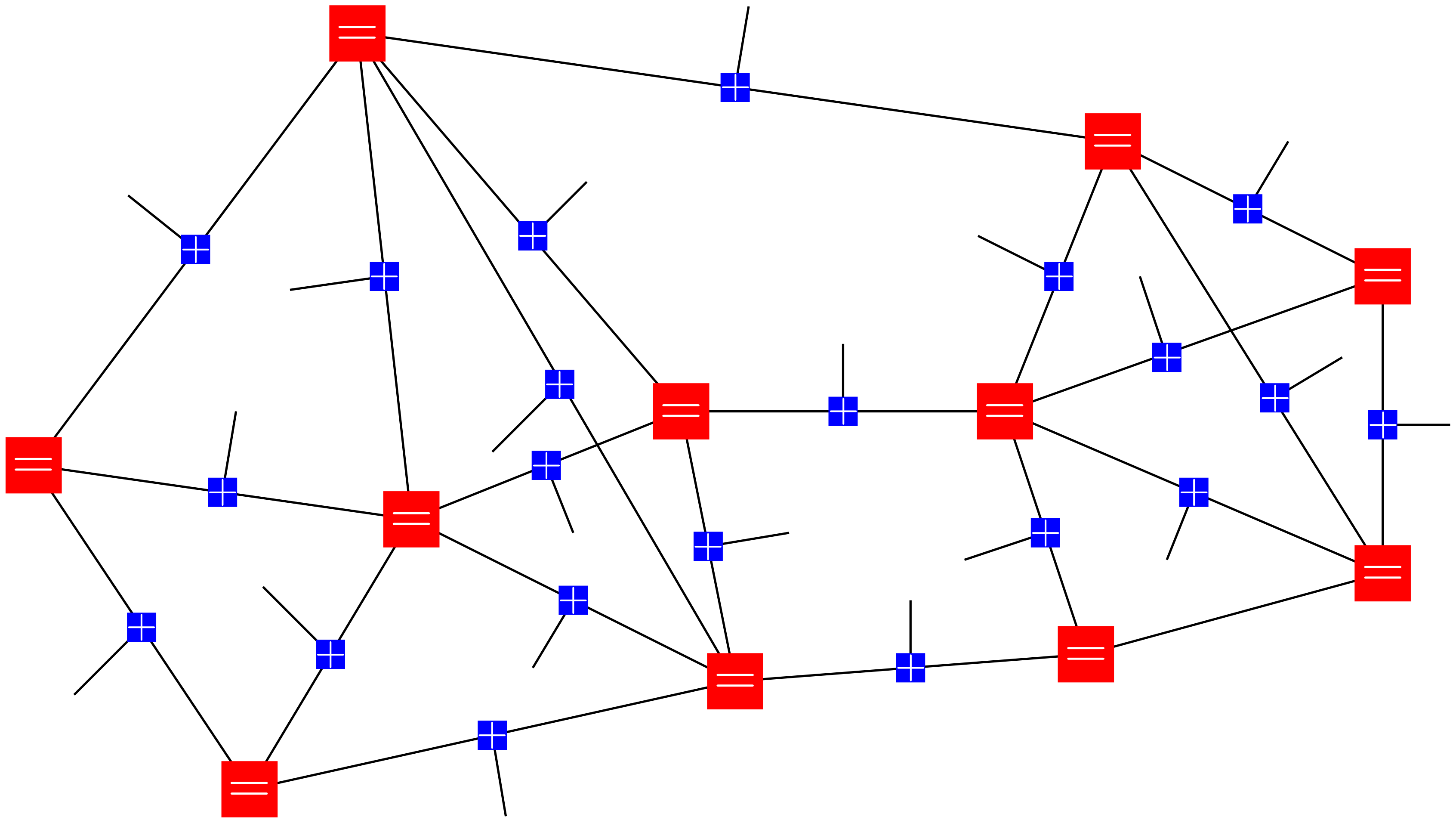

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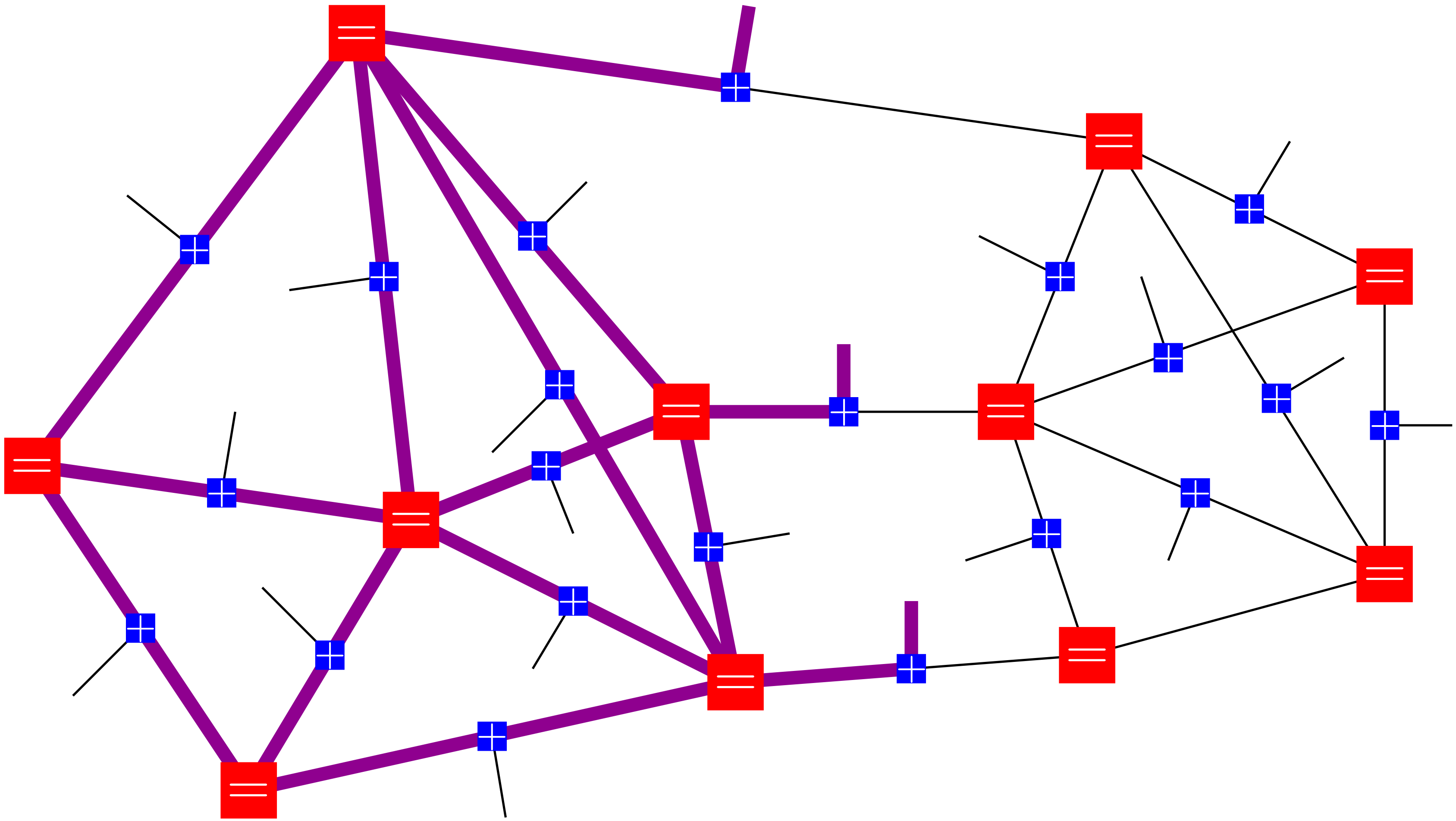

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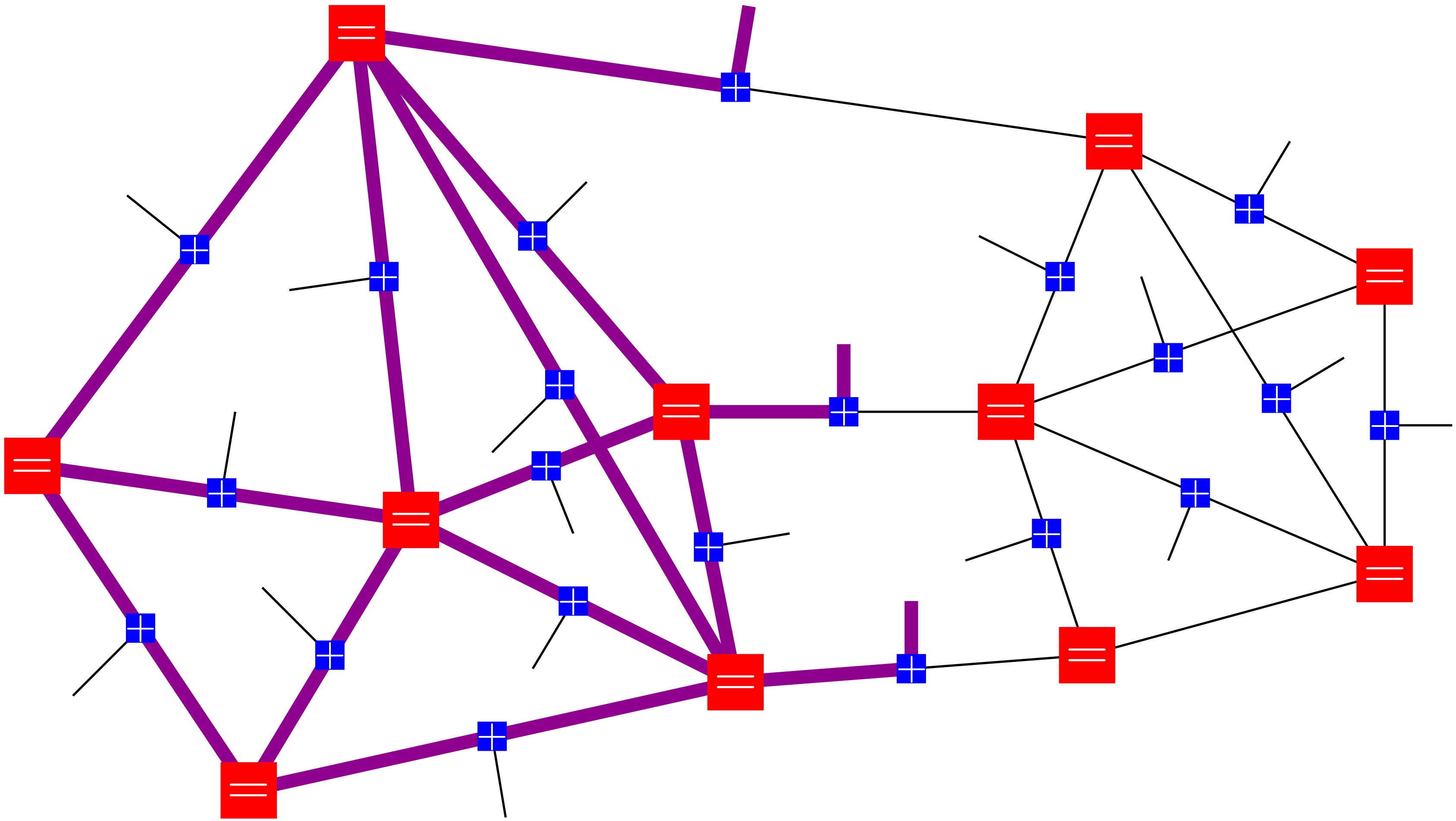
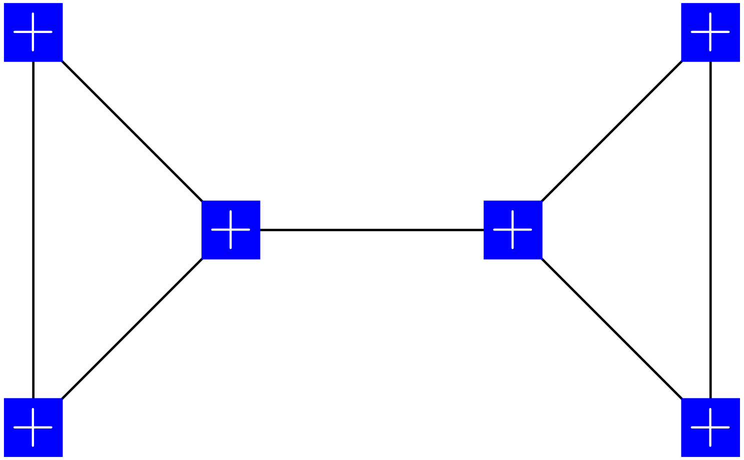


Community detection with the help of MPI algorithms:

[Krzakala, Moore, Mossel, Neeman, Sly, Zdeborová, Zhang, 2013]

Cycle Code NFG vs. Community Detection NFG

It turns out that the cycle code normal factor graph and the community detection normal factor graph are dual to each other in the sense that the sets of valid configurations are given by dual normal factor graphs, cf. NFG duality in [Forney, 2001].



cycle code
normal factor graph

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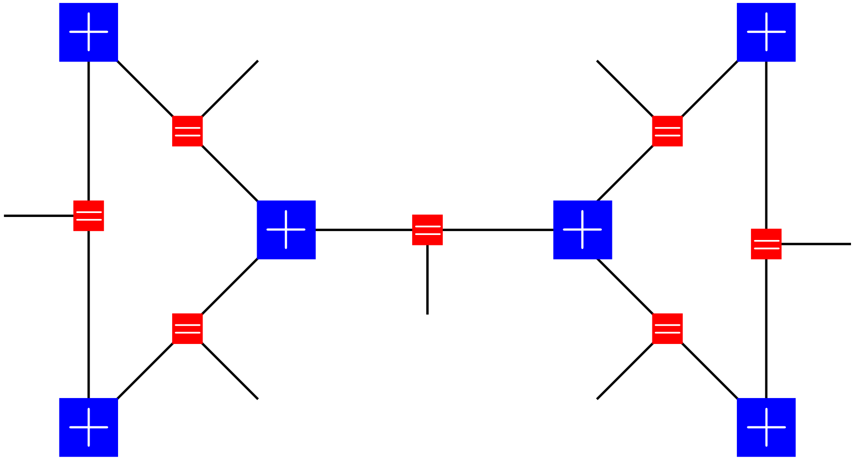

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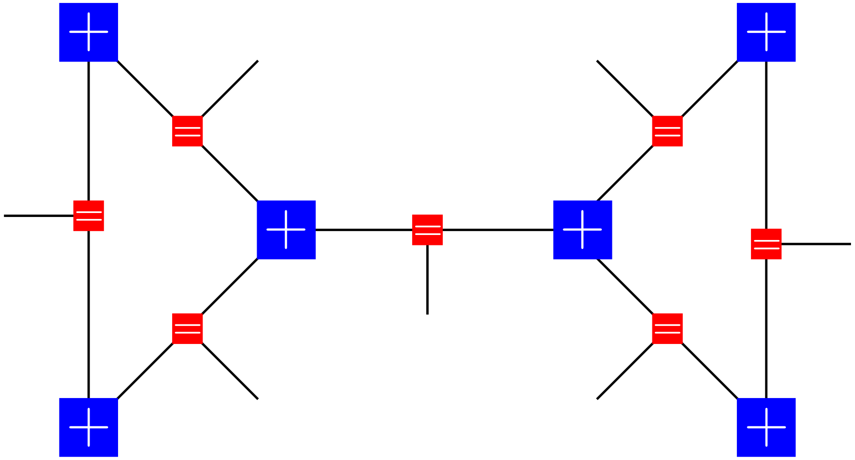

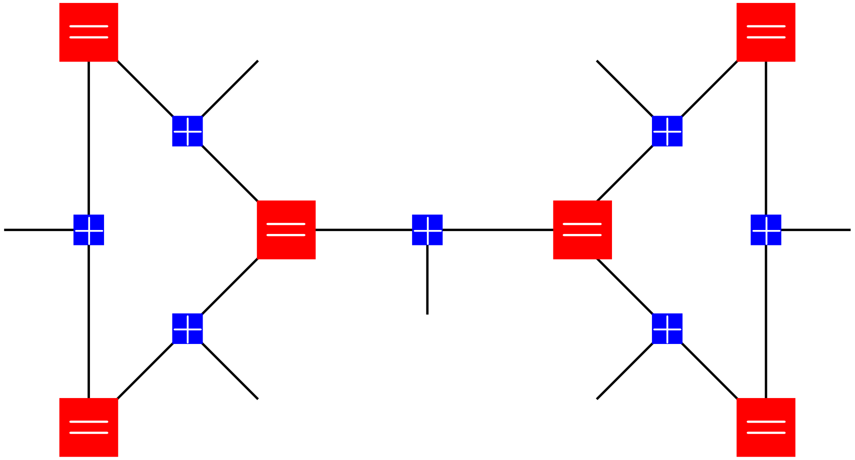

cycle code
normal factor graph

○—●

community detection
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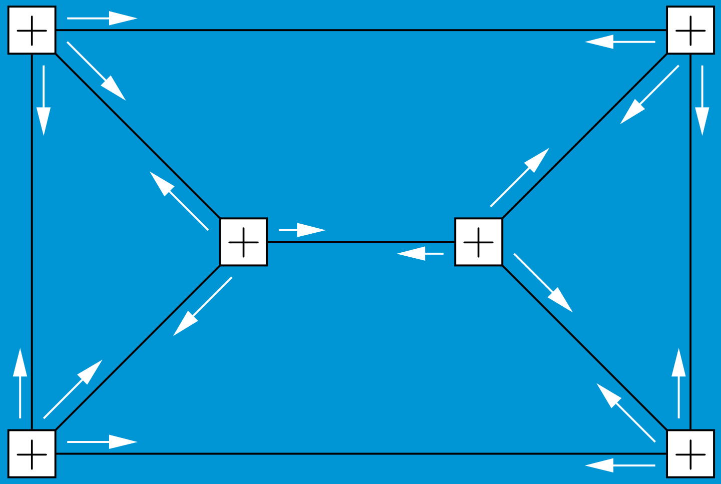

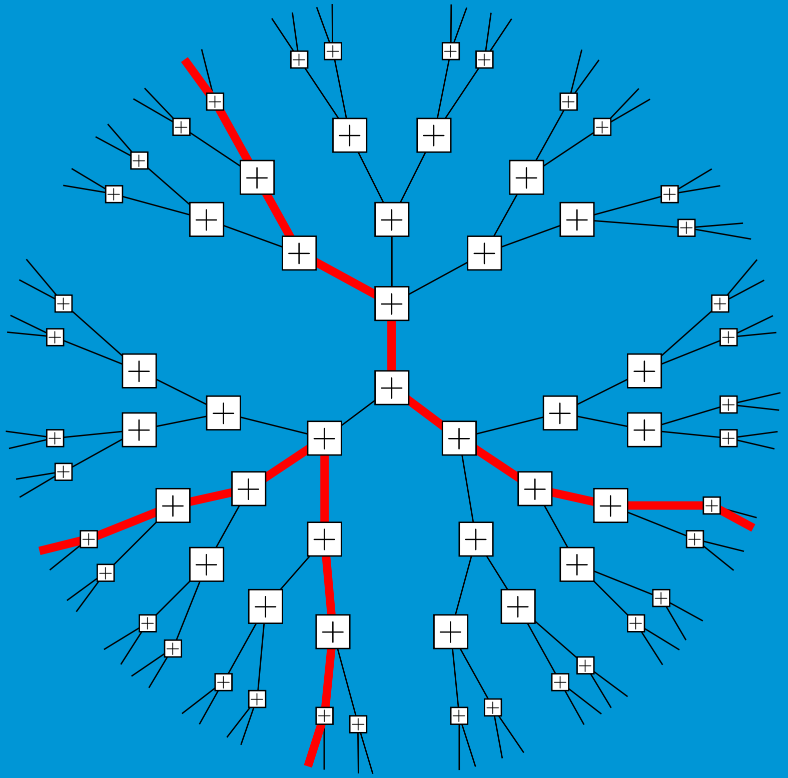

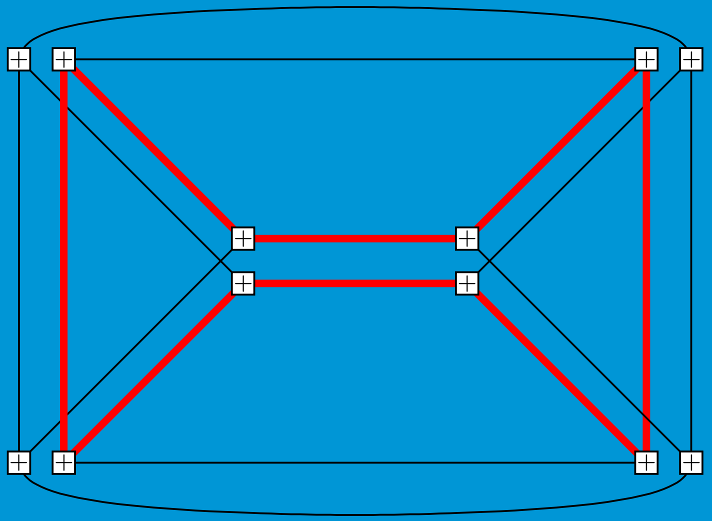

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What Can a Power Series Do For You?

Consider the power series $\zeta(\mathbf{V})$:

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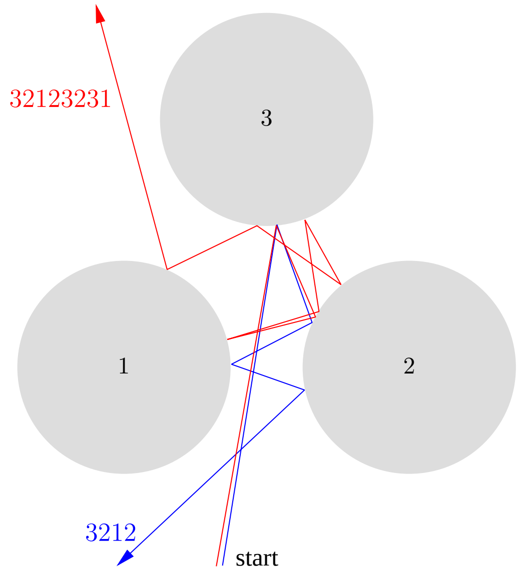
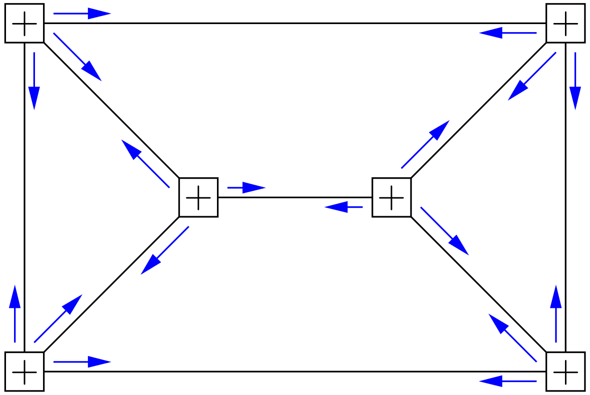
We can obtain useful information from

- ...the expon. vecs. of $\zeta(\mathbf{V})$ [Koetter, Li, V., Walker, 2004/2007]
- ...the coefficients of $\zeta(\mathbf{V})$ [V., 2009/2010] [today]
- ... the evaluation of $\zeta(\mathbf{V})$ for some $\mathbf{V}$ [Watanabe, 2009/2010]
- ...the convergence region of $\zeta(\mathbf{V})$ [today]
- ...

Use of zeta functions for analyzing graphical models.

Analogy Pinball vs. MPI Decoding

 pinball	 message-passing iterative decoding of cycle codes
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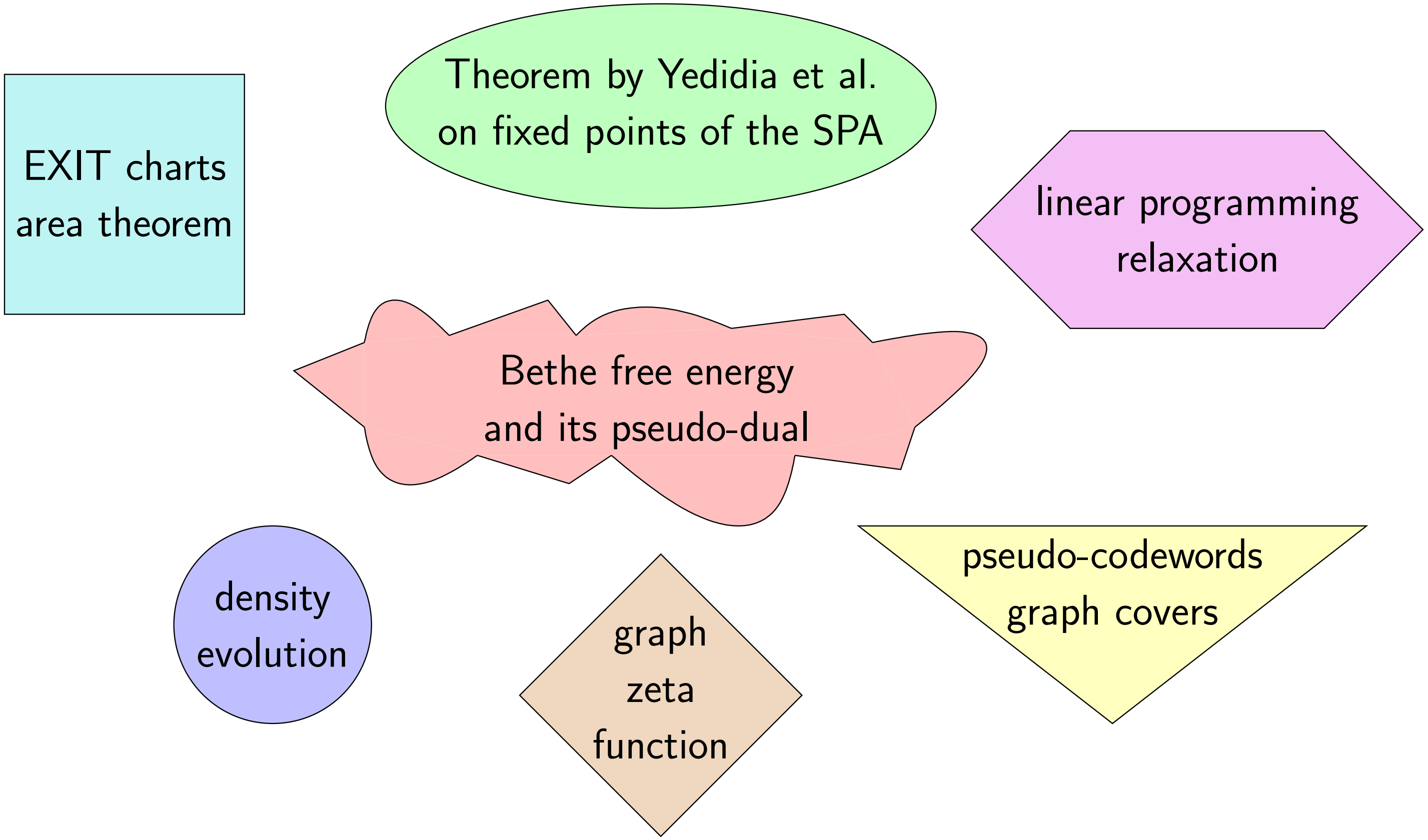
Bethe Free Energy Function

Some of the properties of the Bethe free energy function of the cycle code normal factor graph:

- The induced Bethe free energy function is a convex.
- The sum-product algorithm finds its minimum.



Comments

- We have some generalizations of the above results to general LDPC codes under attenuated SPA decoding.
- Note that [Watanabe, 2010] connects zeta function values to the Hessian of the Bethe free energy function for general factor graphs.
- Use of other concepts from chaos theory for understanding graphical models:
  - Agrawal and Vardy, "The turbo decoding algorithm and its phase trajectories," IEEE Trans. Inf. Theory, 2001.
  - Kocarev, Lehmann, Maggio, Scanavino, Tasev, and Vardy, "Nonlinear dynamics of iterative decoding systems: analysis and applications," IEEE Trans. Inf. Theory, 2006.
  - ...

References

H. D. Pfister and P. O. Vontobel, “On the relevance of graph covers and zeta functions for the analysis of SPA decoding of cycle codes,” Proc. ISIT 2013. (A longer version of this paper is in preparation.)

This work uses and extends results from:

- P. O. Vontobel, “Counting in graph covers: a combinatorial characterization of the Bethe entropy function,” IEEE Trans. Inf. Theory, vol. 59, no 9, pp. 6018–6048, Sep. 2013.
- P. O. Vontobel, “The Bethe permanent of a non-negative matrix,” IEEE Trans. Inf. Theory, vol. 59, no. 3, pp. 1866–1901, Mar. 2013.

Thank you!